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Variational approaches to the elasticity of deformable strings with and without mass redistribution

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The catenary is a curve that has played a significant role in the history of mathematics, finding applications in various disciplines such as mechanics, technology, architecture, the arts, and biology. In this paper, we introduce some generalizations by applying the variational method to deformable strings. We explore two specific cases: (i) in the first case, we investigate the nonlinear behavior of an elastic string with variable length, dependent on the applied boundary conditions; specifically, this analysis serves to introduce the variational method and demonstrate the process of finding analytical solutions; (ii) in the second case, we examine a deformable string with a constant length; however, we introduce mass redistribution within the string through nonlinear elastic interactions. In the first scenario, the deformation state of the string always describes elongation, as compression states prove to be unstable for fully flexible strings. In contrast, in the second scenario, the finite length constraint induces compressive states in specific configurations and regions of the string. However, it is worth noting that the solution to this problem exists only for values of the elastic constant that are not too low, a phenomenon that is studied in detail. We conduct here both analytical and graphical analyses of various geometries, comparing the elastic behavior of the two aforementioned types of strings. Understanding the elastic behavior of deformable strings, especially the second type involving mass redistribution, is crucial for enhancing comprehension in the study of biological filaments or fibers and soft matter. For instance, these investigations can contribute to understanding the mechanisms employed by cells to sense gravity or other mechanical conditions.

1 | INTRODUCTION

The classical problem pertaining to the elasticity of wires and strings is exemplified by the so-called catenary. This mathematical model describes the shape of an inextensible wire, cable, or rope suspended from two ends and subjected to the force of gravity. Apparently, the initial exploration of the catenary was conducted by Leonardo da Vinci (1452–1519), who depicted the form of a rope tethered at its ends in his notebooks [1]. In 1638, Galileo Galilei (1564–1642) tackled the catenary problem. In the second day of “Discorsi e dimostrazioni matematiche intorno a due nuove scienze attinenti alla meccanica e i movimenti locali,” he asserted, through the voice of Salviati, that the catenary assumes the shape of a parabola [2]: “Drive two nails into a wall at a convenient height and at the same level.... Over these two nails hang a light chain.... This chain will assume the form of a parabola....”. Indeed, from this passage, it appears that Galileo equates the shape of a catenary with that of a parabola. Subsequently, this error continued to be unjustly attributed to him by numerous historians and scientists. In fact, in the fourth day, Galileo, once again conveyed through the voice of Salviati, explicitly asserts that the parabola is only an approximation of the catenary. He emphasizes that although the two curves appear similar, they are fundamentally distinct [3]: “But more I want to tell you....that the rope thus stretched,...., bends in lines, which almost approach parabolas....”. Galileo also noted that this approximation becomes more accurate as the curvature decreases and is nearly exact when the elevation is less than 45° [4]. Christiaan Huygens (1629–1695) likely became acquainted with the problem of the hanging chain through his tutor and engaged in extensive correspondence with Marin Mersenne (1588–1648) on this matter. In these letters, Huygens demonstrated to Mersenne that the hanging chain did not assume the shape of a parabola [5]. Indeed, we now understand that the curve formed by such a chain is a catenary or a hyperbolic cosine. The term catenary is the English rendition of the Latin term “catenaria,” a term originally coined by Huygens himself in a letter to Gottfried Leibniz (1646–1716) in November 1690 [5]. Huygens also delved into an additional problem that he had previously discussed with Mersenne–that of forcing the chain to adopt the configuration of a parabola. To continue this story, in the May 1690 edition of Acta Eruditorum, Jacob Bernoulli (1655–1705) presented a solution to the isochrone problem, addressing the definition of a curve along which a body will fall in the same amount of time from any initial position [6]. After this solution, he wrote [7]: “And now let this problem be proposed: to find the curve assumed by a loose string hung freely from two fixed points. And let this string be flexible and of uniform cross-section”. A first response can be found in the Huygens’s paper in Acta Eruditorum [8], which, despite its brevity of less than two pages, was replete with numerous insights into the catenary. While Huygens did not provide a formula for the catenary curve, his compilation of results demonstrates a profound understanding of this curve [9]. This work was immediately followed by the responses of Johann Bernoulli (1667–1748) [10] and Gottfried Leibniz [11], always in Acta Eruditorum [12]. Today, Huygens, Johann Bernoulli, and Leibniz are regarded as the three solvers of the catenary problem [13]. Christian Huygens’ development relied on geometrical arguments, whereas Gottfried Leibniz and Johann Bernoulli utilized the new techniques of differential calculus in their respective demonstrations. Anyway, there was absolutely no mention of hyperbolic functions or any other explicit function in the solutions provided in 1691 [14]. A slightly later reference to the catenary curve can be attributed to Leonhard Euler (1707–1783) in his work on the calculus of variations. In his 1744 publication “Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes,” [15] Euler showed that a catenary rotating about its axis (forming a catenoid) generates the unique minimal surface of revolution. It is interesting to mention that the expressions $(e^x + e^{-x})/2$ and $(e^x - e^{-x})/2$ appear for the first time in Volume I of Euler’s “Introductio in analysin infinitorum” in 1748 [16]. However, Euler did not employ the term “hyperbolic” when referring to these expressions, nor did he introduce any specific notation or nomenclature for them [14]. Today, the analytical examination of the catenary represents a classic topic in physics and mathematics, encountered by every college student during their studies. At the intermediate level, several texts derive the catenary equation without relying on the calculus of variations [17, 18]. The classical treatment is founded on the solution of a differential equation that represents the equilibrium of forces. This is the classical method adopted to investigate the behavior of elastic structures [19–21]. In more advanced mathematics and physics textbooks, the catenary equation is often presented as an application of the calculus of variations, with limited discussion of the underlying physics [22–24]. Some more recent research has reexamined the catenary problem in order to propose alternative solutions of some simplicity or elegance [25–27]. Of course, the catenary problem has always been addressed and solved for a uniform gravitational field, as is always done in elasticity theory problems. However, it is important to mention that recently the problem has also been solved for a catenary immersed in a true central gravitational field, that is, in a potential of the type $-1/r$ [28, 29]. The authors of this study call it “catenaria vera,” that is, true catenary. Interestingly enough, we remark that the hyperbolic cosine reappears in the explicit formula of the true catenary. Another interesting generalization concerns the viscous catenary, which consists in a filament of a highly viscous incompressible fluid, supported at its ends, relaxing under the influence of gravity [30–35]. This matter has also been investigated for filaments composed of soft materials, described by an arbitrary constitutive equation [36], or exhibiting a viscoelastic behavior [37, 38]. More theoretical research has involved the variational approaches to strain energy [39, 40], the analysis of multiple equilibrium states [41, 42], and the introduction of a bending energy [43–45].

The study of the catenary and its variations and generalizations has found applications in various fields. In technology, it has facilitated the examination of the mechanics and configuration of cables and ropes, proving valuable for the stability of structures and the transmission of electricity, notably in railway systems and fabrication processes [46–54]. When

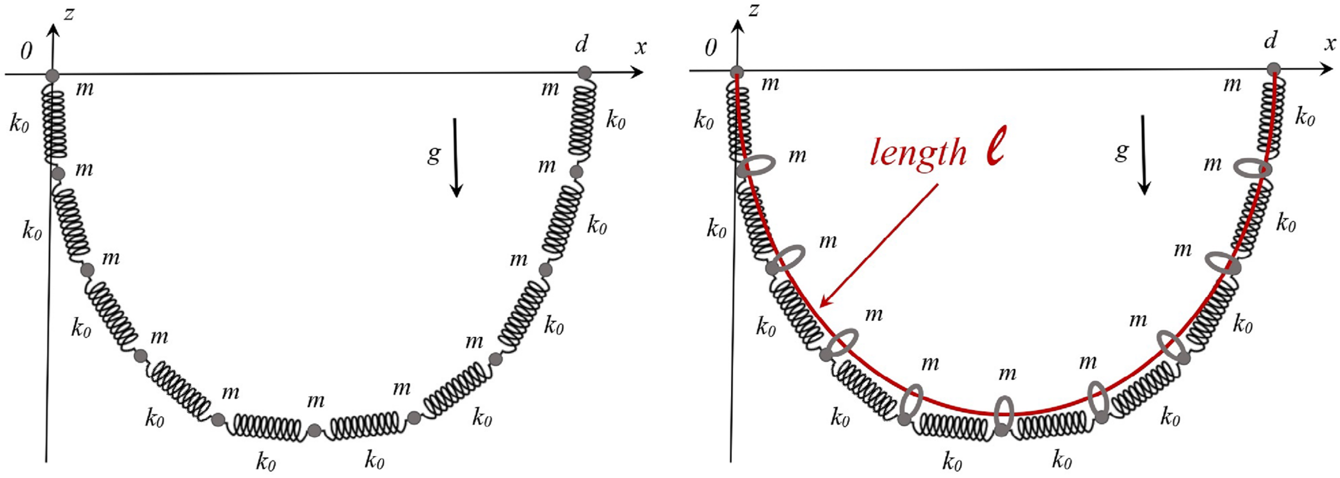

FIGURE 1 Schemes of the discrete version of the strings discussed in this work. Left panel: deformable string with variable length. Right panel: string with fixed length ℓ, exhibiting mass redistribution under applied fields, such as gravity.

inverted from the vertical, the catenary generates the shape of an arch widely employed in architecture. Indeed, In 1675, Robert Hooke (1635–1703) discovered that a thin arch supporting its own weight with pure compression must adopt the inverted shape of a hanging chain, specifically, an inverted catenary [55]. This principle has been applied in numerous architectural works, including notable examples such as the Gateway Arch in St. Louis, constructed in 1963 by the architect Eero Saarinen, the arches in Casa Mila built between 1906 and 1913 by Antoni Gaudí in Barcelona, and the Sagrada Familia, under construction since 1882, representing the pinnacle of Gaudí's work [56]. It is worth noting that the application of catenaries extends to the realm of arts as well. [57]. The geometry and physics of inverse catenary are also helpful in explaining the fountain chain phenomenon, according to which a chain in gravity spontaneously rises from the container containing it before falling to the ground [58–60]. Furthermore, the inverted catenary has also been employed to elucidate the gliding phenomenon, a form of locomotion utilized by various bacteria, including myxobacteria and cyanobacteria [61]. This process is facilitated by a filament extruded from the bacteria, making contact with the substrate and inducing shock propagation [62]. Other applications of catenaries to biology include the examination of plant root tip outlines [63], and the development of growth models [64]. Moreover, the study of simple elastic systems under gravity is useful to better understand the role of biological structures, for example, the cytoskeleton, in the process of sensing gravity by cells [65–67].

The aim of the present paper is to present a variational approach to the mechanics and elasticity of strings useful to study two different generalizations of the catenary problem. The first one deals with the elastic or deformable catenary, which means that we take into consideration possible elongations and compressions of the string (we will see that the compressed state is unstable in this case). Therefore, depending on the elastic constant of the string and on its linear density, the shape of the catenary can be strongly deformed with respect to the classical inextensible catenary. The corresponding system, from the discrete point of view, can be conceptualized through a fully flexible mass-spring chain, where the spring have a finite equilibrium length (see left panel of Figure 1). This is a typical nonlinear elasticity problem, for which we determine the exact equation for an arbitrary dynamic regime, although solutions will be found only in the static cases. Although this problem is a simple generalization of the classical catenary, it remains useful for introducing the variational approach and showing how the exact solutions can be found. It thus provides the basis for the following analysis. The second problem concerns a new soft matter catenary characterized by its inextensibility and the possibility of having mass redistribution within it. From the discrete point of view, it can be conceptualized by means of an inextensible weightless wire along which a series of rings (with mass) connected by elastic springs of finite length can move (see right panel of Figure 1). In the case of such a chain with the ends fixed in gravity, the weight will not be able to change the total length, but it will facilitate the rings to move toward the point of minimum, thus changing the shape of the chain compared to the classical inextensible catenary (compression states can be observed in this case). This original model is then able to describe behaviors of soft matter catenaries where the mass can redistribute over the fixed length according to the applied fields. It is interesting to note that the solution to this problem exists only if the elastic modulus of the chain is larger than a given threshold. In fact, we show that there is a value of the elastic modulus below which the solution is no longer represented by a smooth curve and loses its physical meaning. Indeed, if the elastic modulus is too low, the masses can collapse towards the minimum potential zone making the configuration unattainable. In other words, for too low values of the elastic constant, the solution of the elastic problem is incompatible with the finite length constraint. This point is studied thoroughly, including with a discrete case described in the Appendix. A detailed comparison between the two proposed models is finally discussed.

2 THE VARIATIONAL FORMALISM FOR FLEXIBLE DEFORMABLE STRINGS

We introduce here a method based on a discrete to continuum limit in order to obtain the equation of motion of a fully flexible deformable string. We start with a first approach based on the Lagrangian equations of motion of the analytical mechanics. Then, we use a variational formalism to study the boundary conditions that must be associated to the evolution equation. The system under consideration consists in a mass-spring chain where each nonlinear spring is described by an elastic constant k_0 and an equilibrium length $\Delta\ell$. We note that the spring turns out to be linear only when $\Delta\ell = 0$, a condition that is not interesting for the purposes of this work. Indeed, we want to preserve a finite length of the wire in the continuous limit. We do not specify for the moment the boundary conditions concerning the first and the last mass of the chain. The elastic energy of a single spring can be written as

$$U_{spring} = \frac{1}{2}k_0\left[\left\|\vec{r}_{q+1} - \vec{r}_q\right\| - \Delta\ell\right]^2, \tag{1}$$

where $\vec{r}_1, \dots, \vec{r}_N$ represent the position vectors of the masses of the chain. By means of these premises, we can introduce the Lagrangian L of the system, which depends on position vectors and velocities of the elements of the chain

$$L = L\left(\vec{r}_1, \dots, \vec{r}_N, \frac{d\vec{r}_1}{dt}, \dots, \frac{d\vec{r}_N}{dt}, t\right). \tag{2}$$

More explicitly, it is composed by the kinetic energy and the total potential energy of the system, as it follows

$$L = \sum_{q=1}^{N} \frac{1}{2}m\frac{d\vec{r}_q}{dt}\cdot\frac{d\vec{r}_q}{dt} - \sum_{q=1}^{N-1}\frac{1}{2}k_0\left[\left\|\vec{r}_{q+1} - \vec{r}_q\right\| - \Delta\ell\right]^2 - U_{ext}\left(\vec{r}_1, \dots, \vec{r}_N, t\right). \tag{3}$$

Here, we also introduced an external potential energy $U_{ext}\left(\vec{r}_1, \dots, \vec{r}_N, t\right) = \sum_{q=1}^{N} U\left(\vec{r}_q, t\right)$ representing the external forces applied to the chain. For the sake of simplicity, we assume that all particles are subjected to the same potential energy $U\left(\vec{r}_q, t\right)$, which only depends on $\vec{r}_q$ and time t. By means of the Lagrangian function, we can write the motion equations in the classical form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\vec{r}}_k} - \frac{\partial L}{\partial \vec{r}_k} = 0. \tag{4}$$

To develop the explicit form of the motion equations, we have to perform the following derivatives with respect to $\dot{\vec{r}}_k$

$$\frac{\partial L}{\partial \dot{\vec{r}}_k} = m\dot{\vec{r}}_k, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\vec{r}}_k} = m\ddot{\vec{r}}_k, \tag{5}$$

and those concerning $\vec{r}_k$

$$\frac{\partial L}{\partial \vec{r}_k} = k_0\left[1 - \frac{\Delta\ell}{\left\|\vec{r}_{k+1} - \vec{r}_k\right\|}\right]\left(\vec{r}_{k+1} - \vec{r}_k\right) - k_0\left[1 - \frac{\Delta\ell}{\left\|\vec{r}_k - \vec{r}_{k-1}\right\|}\right]\left(\vec{r}_k - \vec{r}_{k-1}\right) - \frac{\partial U_{ext}\left(\vec{r}_1, \dots, \vec{r}_N, t\right)}{\partial \vec{r}_k}. \tag{6}$$

Summing up, we have the dynamical equations for our system in the form

$$\ddot{\vec{r}}_k = \frac{k_0}{m}\left[1 - \frac{\Delta\ell}{\left\|\vec{r}_{k+1} - \vec{r}_k\right\|}\right]\left(\vec{r}_{k+1} - \vec{r}_k\right) - \frac{k_0}{m}\left[1 - \frac{\Delta\ell}{\left\|\vec{r}_k - \vec{r}_{k-1}\right\|}\right]\left(\vec{r}_k - \vec{r}_{k-1}\right) - \frac{1}{m}\frac{\partial U\left(\vec{r}_k, t\right)}{\partial \vec{r}_k}, \tag{7}$$

since $\partial U_{ext}\left(\vec{r}_1, \ldots, \vec{r}_N, t\right)/\partial \vec{r}_k = \partial U\left(\vec{r}_k, t\right)/\partial \vec{r}_k$. We underline that these equations are strongly nonlinear because of the strictly positive equilibrium length of the springs composing the chain. Of course, if $\Delta\ell = 0$, the equations turn out to be linear and they would represent the discrete version of the D'Alambert wave equation. This nonlinearity plays a central role in the following developments. That being said, we can now perform the continuum limit in order to introduce the equation for a one-dimensional elastic string. The unperturbed or unstretched string has the total length ℓ and therefore we can write $(N-1)\Delta\ell = \ell$ with $N \to \infty$ and $\Delta\ell \to 0$. In the continuum limit, the variables $\vec{r}_k(t)$ will be substituted with a function $\vec{r}(s,t)$, where s is the curvilinear abscissa, with $s \in [0, \ell]$. It means that s introduces a natural parametrization for the curve representing the chain only in the unstretched (not elongated and not compressed) condition. We can therefore add that the function $\vec{r}(s,t)$ represents the position of the point of the arbitrarily deformed chain having curvilinear abscissa s in the unstretched chain. Moreover, $\Delta\ell$ represents the increment of the variable s since $(N-1)\Delta\ell = \ell$ and $s \in [0, \ell]$. Therefore, we can rewrite Equation (7) as

$$\frac{m}{\Delta\ell}\ddot{\vec{r}}_k = k_0\Delta\ell \frac{\left[1 - \frac{\Delta\ell}{\|\vec{r}_{k+1}-\vec{r}_k\|}\right]\frac{\vec{r}_{k+1}-\vec{r}_k}{\Delta\ell} - \left[1 - \frac{\Delta\ell}{\|\vec{r}_k-\vec{r}_{k-1}\|}\right]\frac{\vec{r}_k-\vec{r}_{k-1}}{\Delta\ell}}{\Delta\ell} - \frac{1}{\Delta\ell}\frac{\partial U\left(\vec{r}_k, t\right)}{\partial \vec{r}_k}, \tag{8}$$

where we can easily identify the first order and second order incremental ratios. We can define the mass density (for unit length) $\lambda = \lim_{N\to\infty} m/\Delta\ell$ (with $m \to 0$ and $\Delta\ell \to 0$), and the density of potential energy (for unit length) $\mathcal{U}(\vec{r}, t) = \lim_{N\to\infty} U\left(\vec{r}_k, t\right)/\Delta\ell$ (with $U \to 0$ and $\Delta\ell \to 0$). Moreover, for any single spring, we can write $f = k_0(\Delta\ell' - \Delta\ell)$, where f is the applied force, $\Delta\ell'$ is the deformed length, and $\Delta\ell$ is the original length. If this spring represents a small portion of wire, we can also write $\sigma = E\varepsilon$, where $\sigma = f/S$ is the stress, S is the cross-sectional area, and $\varepsilon = (\Delta\ell' - \Delta\ell)/\Delta\ell$ is the strain. Comparing these two points of view, we find that $k_0\Delta\ell = SE$, or more precisely, $SE = \lim_{N\to\infty} k_0\Delta\ell$, with $k_0 \to \infty$ and $\Delta\ell \to 0$. It means that we are considering engineering stress and strain. As usual, for not too large deformations, the results should not deviate too much from those obtained with true stress and strain. To conclude, the continuum limit of Equation (8) can be written as

$$\lambda\frac{\partial^2\vec{r}}{\partial t^2} = SE\frac{\partial}{\partial s}\left\{\left[1 - \frac{1}{\left\|\frac{\partial\vec{r}}{\partial s}\right\|}\right]\frac{\partial\vec{r}}{\partial s}\right\} - \frac{\partial\mathcal{U}}{\partial\vec{r}}, \tag{9}$$

with the unknown vector $\vec{r}(s,t)$. This equation fully describes the statics and the dynamics of the elastic string or wire. However, for the moment, we do not know the form of boundary conditions to be imposed in order to find solutions to real problems. To solve this difficulty, it is better to reformulate the procedure through a variational approach. To this aim, we can rewrite the Lagrangian function in Equation (3) as it follows

$$L = \sum_{q=1}^{N}\left[\frac{m}{2\Delta\ell}\frac{d\vec{r}_q}{dt}\cdot\frac{d\vec{r}_q}{dt}\right]\Delta\ell - \sum_{q=1}^{N-1}\left[\frac{k_0\Delta\ell}{2}\left(\frac{\|\vec{r}_{q+1}-\vec{r}_q\|}{\Delta\ell} - 1\right)^2\right]\Delta\ell - \sum_{q=1}^{N}\left[\frac{U\left(\vec{r}_k, t\right)}{\Delta\ell}\right]\Delta\ell. \tag{10}$$

In this expression we can observe, in the limit of $\Delta\ell \to 0$, the convergence of the sums to integrals, eventually obtaining

$$L = \int_0^\ell \frac{1}{2}\lambda\left(\frac{\partial\vec{r}}{\partial t}\right)^2 ds - \int_0^\ell \frac{1}{2}SE\left[\left\|\frac{\partial\vec{r}}{\partial s}\right\| - 1\right]^2 ds - \int_0^\ell \mathcal{U}(\vec{r})ds. \tag{11}$$

This result allows us to define a Lagrangian density $\mathcal{L}$ by means of the expression

$$\mathcal{L} = \frac{1}{2}\lambda\left(\frac{\partial\vec{r}}{\partial t}\right)^2 - \frac{1}{2}SE\left[\left\|\frac{\partial\vec{r}}{\partial s}\right\| - 1\right]^2 - \mathcal{U}(\vec{r}), \tag{12}$$

such that $L = \int_0^\ell \mathcal{L}ds$. These quantities are of central importance in introducing the action $\mathcal{S}$ of the system as

$$\mathcal{S} = \int_0^T Ldt = \int_0^T\int_0^\ell \mathcal{L}dsdt. \tag{13}$$

The evolution of the system is then based on the stationary-action principle (or the principle of least action) stating that the evolution of the system makes the action functional stationary (or minimal). The calculus of variations allows us to study the stationarity conditions of the action defined in Equation (13). Let $\vec{h}(s,t)$ be a variation allowed to the solution $\vec{r}(s,t)$. We can calculate the perturbation to the action induced by this variation h. We find

$$\mathcal{S}(\vec{r}+\alpha\vec{h}) = \int_0^T\int_0^\ell \frac{1}{2}\lambda\left(\frac{\partial\vec{r}}{\partial t}+\alpha\frac{\partial\vec{h}}{\partial t}\right)^2 dsdt - \int_0^T\int_0^\ell \frac{1}{2}SE\left[\left\|\frac{\partial\vec{r}}{\partial s}+\alpha\frac{\partial\vec{h}}{\partial s}\right\|-1\right]^2 dsdt - \int_0^T\int_0^\ell \mathcal{U}(\vec{r}+\alpha\vec{h})dsdt, \quad (14)$$

where α is a real parameter. In order to calculate the Gateaux differential of the action $\mathcal{S}$, we firstly evaluate $\partial\mathcal{S}(\vec{r}+\alpha\vec{h})/\partial\alpha$

$$\frac{\partial\mathcal{S}(\vec{r}+\alpha\vec{h})}{\partial\alpha} = \int_0^T\int_0^\ell \lambda\left(\frac{\partial\vec{r}}{\partial t}+\alpha\frac{\partial\vec{h}}{\partial t}\right)\cdot\frac{\partial\vec{h}}{\partial t}dsdt - \int_0^T\int_0^\ell SE\left[\left\|\frac{\partial\vec{r}}{\partial s}+\alpha\frac{\partial\vec{h}}{\partial s}\right\|-1\right]\frac{\left(\frac{\partial\vec{r}}{\partial s}+\alpha\frac{\partial\vec{h}}{\partial s}\right)\cdot\frac{\partial\vec{h}}{\partial s}}{\left\|\frac{\partial\vec{r}}{\partial s}+\alpha\frac{\partial\vec{h}}{\partial s}\right\|}dsdt$$

$$-\int_0^T\int_0^\ell \frac{\partial\mathcal{U}}{\partial\vec{r}}\cdot\vec{h}dsdt, \quad (15)$$

and we then specialize the result for $\alpha = 0$

$$\left.\frac{\partial\mathcal{S}(\vec{r}+\alpha\vec{h})}{\partial\alpha}\right|_{\alpha=0} = \int_0^T\int_0^\ell\left[\lambda\frac{\partial\vec{r}}{\partial t}\cdot\frac{\partial\vec{h}}{\partial t} - SE\left(1-\frac{1}{\left\|\frac{\partial\vec{r}}{\partial s}\right\|}\right)\frac{\partial\vec{r}}{\partial s}\cdot\frac{\partial\vec{h}}{\partial s} - \frac{\partial\mathcal{U}}{\partial\vec{r}}\cdot\vec{h}\right]dsdt. \quad (16)$$

As usual, one can work out the first and second terms by integration by parts, as follows

$$\left.\frac{\partial\mathcal{S}(\vec{r}+\alpha\vec{h})}{\partial\alpha}\right|_{\alpha=0} = \int_0^T\int_0^\ell\left\{-\lambda\frac{\partial^2\vec{r}}{\partial t^2} + SE\frac{\partial}{\partial s}\left[\left(1-\frac{1}{\left\|\frac{\partial\vec{r}}{\partial s}\right\|}\right)\frac{\partial\vec{r}}{\partial s}\right] - \frac{\partial\mathcal{U}}{\partial\vec{r}}\right\}\cdot\vec{h}dsdt$$

$$+\int_0^\ell \lambda\frac{\partial\vec{r}}{\partial t}\cdot\vec{h}ds\bigg|_0^T - \int_0^T SE\left(1-\frac{1}{\left\|\frac{\partial\vec{r}}{\partial s}\right\|}\right)\frac{\partial\vec{r}}{\partial s}\cdot\vec{h}dt\bigg|_0^\ell. \quad (17)$$

This Gateaux differential must be zero for any perturbation $\vec{h}$ compatible with the boundary conditions, and therefore we deduce that the first double integral in the right hand side of Equation (17) yields the equation of motion already stated in Equation (9). The second integral concerns the time initial and/or final conditions and is typical of variational approaches in mechanics. Finally, the third integral is particularly important since generates the space boundary conditions. If the position vectors $\vec{r}(0,t)$ and $\vec{r}(\ell,t)$ of the chain extremities are fixed, we have that $\vec{h}(0,t) = 0$ and $\vec{h}(\ell,t) = 0$, and therefore the third term at the right hand side of Equation (17) is zero. However, if for example $\vec{r}(\ell,t)$ is free to move in the space, then $\vec{h}(\ell,t)$ is also free, and thus we must impose $\left\|\frac{\partial\vec{r}}{\partial s}\right\| = 1$ for $s = \ell$. It is indeed evident that in the free end there must be no local deformation, and this means that s is a local natural parameter near the free end. This explains how the calculus of variations is important in determining not only the equation to be solved but also the conditions to be associated to correctly approach the physical problem. To conclude, we remark that the quantity $\left\|\frac{\partial\vec{r}}{\partial s}\right\|$ can be named stretch and must be always positive for a regular motion of the string [21]. Moreover, where $\left\|\frac{\partial\vec{r}}{\partial s}\right\| > 1$ the string is elongated, and where $\left\|\frac{\partial\vec{r}}{\partial s}\right\| < 1$, the string is compressed. In the following, we will study the three configurations schematized in Figure 2, where the adopted boundary and loading conditions can be clearly identified.

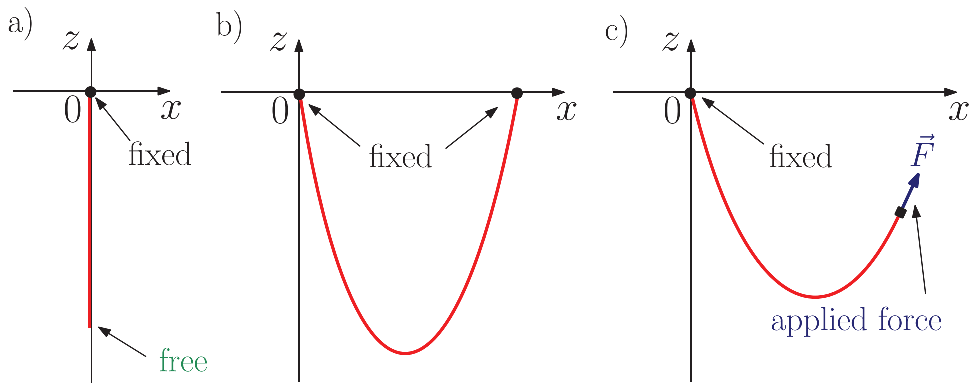

FIGURE 2 Scheme of the three configurations studied in this work: (a) elastic wire hanging form one hand, (b) elastic wire anchored at both ends, (c) elastic wire with one end fixed and the other subjected to applied force. The boundary and loading conditions can be clearly identified for each configuration. These three geometries are studied for both the extensible string (in Section 2) and the inextensible string with mass redistribution (in Section 3).

2.1 | Elastic wire hanging from one end

As a first example of application, we consider an elastic wire hanging at a fixed point and subject to the force of gravity. It means that the first extremity is fixed, say at the origin of the axes, and the second one is free. Of course, the wire is aligned with the z-axis, for $z < 0$ (see Figure 2a). In static conditions, from Equation (9), we have

$$SE\frac{\partial}{\partial s}\left\{\left[1-\frac{1}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right]\frac{\partial \vec{r}}{\partial s}\right\} = \frac{\partial \mathcal{U}}{\partial \vec{r}}, \tag{18}$$

and, moreover, we consider $\mathcal{U} = \lambda g z$, where g is gravitational acceleration. Projecting this equation onto the vertical z-axis, we obtain

$$SE\frac{\partial}{\partial s}\left\{\left[1-\frac{1}{\left|\frac{\partial z}{\partial s}\right|}\right]\frac{\partial z}{\partial s}\right\} = \lambda g. \tag{19}$$

In our case we clearly have that $\partial z/\partial s < 0$. Hence, we get

$$SE\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s}+1\right) = \lambda g. \tag{20}$$

A first integration yields

$$\frac{\partial z}{\partial s}+1 = \frac{\lambda g}{SE}s + C_1. \tag{21}$$

A second integration leads to

$$z(s) = \frac{\lambda g}{2SE}s^2 - s + C_1 s + C_2. \tag{22}$$

The boundary conditions are given by $z(0) = 0$ in the fixed end, and $\left\|\frac{\partial \vec{r}}{\partial s}\right\| = 1$ for $s = \ell$, that is, $\partial z/\partial s = -1$ for $s = \ell$ (free end, see Figure 2a). We easily find that $C_2 = 0$ and $C_1 = -\lambda g \ell/(SE)$, and the final solution is

$$z(s) = \frac{\lambda g}{2SE}s^2 - s - \frac{\lambda g \ell}{SE}s. \tag{23}$$

Therefore, the position of the free extremity is identified by $z(\ell) = -\ell - \lambda g \ell^2/(2SE)$, and the extension of the wire due to its weight is given by $\lambda g \ell^2/(2SE) = Mg\ell/(2SE)$, where $M = \lambda \ell$ is the total mass distributed over the wire. Interestingly,

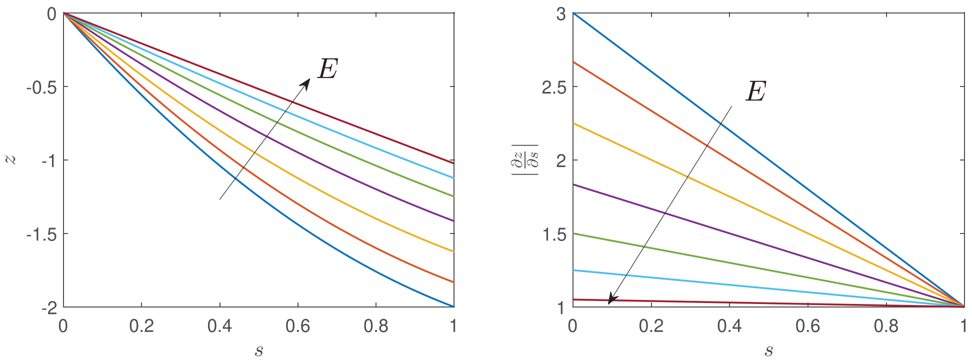

FIGURE 3 Elastic wire hanging from one end. Left panel: plot of the parametrization $z = z(s)$ for different values of the elastic constant E. Right panel: plot of the stretch $\left|\frac{\partial z}{\partial s}\right|$, representing the elongation of the string. We adopted the parameters $\ell = 1$, $\lambda g = 1$, and $ES = 0.5$, 0.6, 0.8, 1.2, 2, 4, 20, in arbitrary units.

the obtained total extension is one half of that corresponding to the case of a weightless wire with a mass M concentrated at the free end, which is given by $Mg\ell/(SE)$. This difference is due to the fact that the continuous mass distribution obviously reduces the total extension of the wire with respect to the concentrated mass. We also observe that when E tends to infinity we get the simple solution $z(s) = -s$ that obviously represents the undeformed string. Concerning the stretch of the string, we have that $\left\|\frac{\partial \vec{r}}{\partial s}\right\| = \left|\frac{\partial z}{\partial s}\right| = 1 + \frac{\lambda g}{SE}(\ell - s)$ and therefore the string is everywhere elongated with the maximum elongation for $s = 0$ (fixed end), and without local elongation for $s = \ell$ (free end).

The results obtained are shown in Figure 3, where the plots of the parametrization $z(s)$ and of the stretch $\left|\frac{\partial z}{\partial s}\right|$ can be found for different values of the elastic constant E of the string. We note the increasing elongation of the curve as E decreases (left panel) and the fact that the string is always in a stretched state, which is largest near the upper constraint and absent at the lower point (right panel). Here and in the following plots, we use arbitrary values of the product ES only to explain the physics of the problem. We recall that the Young's modulus of rubber-like materials ranges between 1/1000 and 1/10 GPa, for metals is around 100 GPa, for a nylon rope between 2 and 4 Gpa, and for a Deoxyribonucleic acid (DNA) macromolecule it is around 350 MPa. For example, if we consider 1 nm for the DNA radius, we get $ES = 1$ nN. Otherwise, a nylon rope with $E = 3$ GPa and a radius of 5 mm has an ES value of 2.5×10^5 N. We then remark that the order of magnitude of ES varies strongly with material and size.

2.2 | Elastic wire anchored at both ends

We consider now a two-dimensional problem concerning an extensible wire with the two extremities tethered at $\vec{r}(0) = (0,0)$ and $\vec{r}(\ell) = (d,0)$, and subjected to the gravity (see Figure 2b). It represents the elastic or deformable catenary. The static equations of the problem are

$$SE\frac{\partial}{\partial s}\left\{\left[1 - \frac{1}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right]\frac{\partial x}{\partial s}\right\} = 0, \qquad SE\frac{\partial}{\partial s}\left\{\left[1 - \frac{1}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right]\frac{\partial z}{\partial s}\right\} = \lambda g. \tag{24}$$

Hence, a first integration delivers

$$\left[1 - \frac{1}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right]\frac{\partial x}{\partial s} = C, \qquad \left[1 - \frac{1}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right]\frac{\partial z}{\partial s} = \frac{\lambda g}{SE}s + D, \tag{25}$$

with arbitrary constants C and D. The constant D can be easily found by symmetry. Indeed, $z(s)$ must have a minimum point for $s = \ell/2$ and, therefore, $\partial z/\partial s$ must be zero for $s = \ell/2$. This implies that $D = -\lambda g\ell/(2SE)$. Now, from

Equation (25), we get

$$\frac{\partial z}{\partial s} = \frac{1}{C}\frac{\partial x}{\partial s}\left(\frac{\lambda g}{SE}s - \frac{\lambda g\ell}{2SE}\right). \tag{26}$$

Then, we can calculate the square of the stretch, as follows

$$\left\|\frac{\partial \vec{r}}{\partial s}\right\|^2 = \left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial s}\right)^2 = \left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial x}{\partial s}\right)^2 \frac{1}{C^2}\left(\frac{\lambda g}{SE}s - \frac{\lambda g\ell}{2SE}\right)^2, \tag{27}$$

and, since $\partial x/\partial s > 0$ in our geometry, we can also write

$$\left\|\frac{\partial \vec{r}}{\partial s}\right\| = \frac{\partial x}{\partial s}\sqrt{1 + \frac{1}{C^2}\left(\frac{\lambda g}{SE}s - \frac{\lambda g\ell}{2SE}\right)^2}. \tag{28}$$

From the first relation in Equation (25), we deduce that

$$\frac{\partial x}{\partial s} = C + \frac{\frac{\partial x}{\partial s}}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|} = C + \frac{1}{\sqrt{1 + \frac{1}{C^2}\left(\frac{\lambda g}{SE}s - \frac{\lambda g\ell}{2SE}\right)^2}}, \tag{29}$$

and by integrating we have

$$x(s) = Cs + \int_0^s \frac{dt}{\sqrt{1 + \frac{1}{C^2}\left(\frac{\lambda g}{SE}t - \frac{\lambda g\ell}{2SE}\right)^2}}. \tag{30}$$

This integral can be performed by the change of variable $\xi = \frac{\lambda g}{SE}t - \frac{\lambda g\ell}{2SE}$, leading to

$$x(s) = Cs + \frac{SE}{\lambda g}\int_{-\frac{\lambda g\ell}{2SE}}^{\frac{\lambda g}{SE}s - \frac{\lambda g\ell}{2SE}} \frac{d\xi}{\sqrt{1 + \frac{\xi^2}{C^2}}}. \tag{31}$$

Since

$$\int \frac{d\xi}{\sqrt{1 + \frac{\xi^2}{C^2}}} = C\ln\left(\frac{\xi}{C} + \sqrt{1 + \frac{\xi^2}{C^2}}\right) + const, \tag{32}$$

we obtain the final solution for $x(s)$ in the form

$$x(s) = Cs + C\frac{SE}{\lambda g}\ln\frac{\frac{1}{C}\left(\frac{\lambda g}{SE}s - \frac{\lambda g\ell}{2SE}\right) + \sqrt{1 + \frac{1}{C^2}\left(\frac{\lambda g}{SE}s - \frac{\lambda g\ell}{2SE}\right)^2}}{-\frac{1}{C}\frac{\lambda g\ell}{2SE} + \sqrt{1 + \frac{1}{C^2}\left(\frac{\lambda g\ell}{2SE}\right)^2}}, \tag{33}$$

where the constant C must be calculated by imposing $x(\ell) = d$. In order to obtain a similar result for $z(s)$, we start by considering Equation (26), from which we can write

$$\frac{\partial z}{\partial s} = \frac{1}{C}\left(C + \frac{1}{\sqrt{1 + \frac{1}{C^2}\left(\frac{\lambda g}{SE}s - \frac{\lambda g\ell}{2SE}\right)^2}}\right)\left(\frac{\lambda g}{SE}s - \frac{\lambda g\ell}{2SE}\right). \tag{34}$$

Again, by integration we obtain

$$z(s)=\frac{\lambda g}{2SE}s^2-\frac{\lambda g\ell}{2SE}s+\frac{1}{C}\int_0^s\frac{\frac{\lambda g}{SE}t-\frac{\lambda g\ell}{2SE}}{\sqrt{1+\frac{1}{C^2}\left(\frac{\lambda g}{SE}t-\frac{\lambda g\ell}{2SE}\right)^2}}dt. \tag{35}$$

As before, the change of variable $\xi=\frac{\lambda g}{SE}t-\frac{\lambda g\ell}{2SE}$, delivers

$$z(s)=\frac{\lambda g}{2SE}s^2-\frac{\lambda g\ell}{2SE}s+\frac{1}{C}\frac{SE}{\lambda g}\int_{-\frac{\lambda g\ell}{2SE}}^{\frac{\lambda g}{SE}s-\frac{\lambda g\ell}{2SE}}\frac{\xi d\xi}{\sqrt{1+\frac{\xi^2}{C^2}}}, \tag{36}$$

and therefore we obtain

$$z(s)=\frac{\lambda g}{2SE}s^2-\frac{\lambda g\ell}{2SE}s+C\frac{SE}{\lambda g}\left[\sqrt{1+\frac{1}{C^2}\left(\frac{\lambda g}{SE}s-\frac{\lambda g\ell}{2SE}\right)^2}-\sqrt{1+\frac{1}{C^2}\left(\frac{\lambda g\ell}{2SE}\right)^2}\right]. \tag{37}$$

Now, Equations (33) and (37) represent the solution of the problem provided that we determine the constant C with the condition $x(\ell)=d$. This condition can be explicitly written in the form

$$d=C\ell+C\frac{SE}{\lambda g}\ln\frac{\frac{1}{C}\frac{\lambda g\ell}{2SE}+\sqrt{1+\frac{1}{C^2}\left(\frac{\lambda g\ell}{2SE}\right)^2}}{-\frac{1}{C}\frac{\lambda g\ell}{2SE}+\sqrt{1+\frac{1}{C^2}\left(\frac{\lambda g\ell}{2SE}\right)^2}}, \tag{38}$$

which can be further simplified by elaborating the term $\ln[(\sqrt{B}+A)/(\sqrt{B}-A)]$, where $A=\frac{1}{C}\frac{\lambda g\ell}{2SE}$ and $B=1+A^2$. We obtain $\ln[(\sqrt{B}+A)/(\sqrt{B}-A)]=2\ln(\sqrt{B}+A)=2\,\mathrm{arcsinh}\,A$. To compact the resulting expressions, instead of considering the constant C, we can introduce the coefficient $\mu=\lambda g/(CSE)$, and it must fulfils the equation

$$\frac{\mu\ell}{2}=\sinh\left(\frac{\mu d}{2}-\frac{\lambda g\ell}{2SE}\right). \tag{39}$$

To conclude, Equations (33) and (37) can be rewritten in terms of the new coefficient μ as follows

$$x(s)=\frac{\lambda g}{\mu SE}\left(s-\frac{\ell}{2}\right)+\frac{d}{2}+\frac{1}{\mu}\ln\left[\mu\left(s-\frac{\ell}{2}\right)+\sqrt{1+\mu^2\left(s-\frac{\ell}{2}\right)^2}\right], \tag{40}$$

$$z(s)=\frac{\lambda g}{2SE}s(s-\ell)+\frac{1}{\mu}\left[\sqrt{1+\mu^2\left(s-\frac{\ell}{2}\right)^2}-\sqrt{1+\mu^2\left(\frac{\ell}{2}\right)^2}\right], \tag{41}$$

where we used the property: $\ln\left[-\mu\ell/2+\sqrt{1+\mu^2(\ell/2)^2}\right]=\mathrm{arcsinh}(-\mu\ell/2)=-\mu d/2+\lambda g\ell/(2SE)$. Then, once the solution of the Equation (39) is found, the value of μ can be used in the parametric representation of the curve given in Equations (40) and (41). Note that s in general is not a natural parameter because the wire is not in an undeformed condition.

If the elastic parameter E approaches infinity, the curve must coincide with the catenary, that is, the curve that an idealized undeformable hanging chain assumes under its own weight when supported only at its ends in a uniform gravitational field. From classical textbooks of mechanics and calculus of variations [17, 21, 22], we learn that such a curve in cartesian coordinates is given by

$$z=\frac{1}{\mu}\left[\cosh\left(\mu x-\frac{\mu d}{2}\right)-\cosh\left(\frac{\mu d}{2}\right)\right], \tag{42}$$

with

$$\frac{\mu\ell}{2}=\sinh\left(\frac{\mu d}{2}\right) \tag{43}$$

where, for ease of comparison, we have used the same geometry and notations adopted in this article. This classical result shows that the shape of the inextensible wire is coincident with a properly rescaled and translated hyperbolic cosine. We try to rewrite this classical solution in parametric form with a natural parameter s. This is useful to draw a comparison with Equations (39), (40), and (41) when E approaches infinity. Given the curve $z=f(x)$ defined in Equation (42), its length in the region $(0,x)$ is given by

$$s(x)=\int_0^x\sqrt{1+f'(t)^2}dt=\int_0^x\cosh\left(\mu t-\frac{\mu d}{2}\right)dt=\frac{1}{\mu}\left[\sinh\left(\mu x-\frac{\mu d}{2}\right)+\sinh\left(\frac{\mu d}{2}\right)\right]. \tag{44}$$

By evaluating the inverse function $x(s)$ of previous expression and by determining $z(s)=f(x(s))$, we get

$$x(s)=\frac{d}{2}+\frac{1}{\mu}\ln\left[\mu\left(s-\frac{\ell}{2}\right)+\sqrt{1+\mu^2\left(s-\frac{\ell}{2}\right)^2}\right], \tag{45}$$

$$z(s)=\frac{1}{\mu}\left[\sqrt{1+\mu^2\left(s-\frac{\ell}{2}\right)^2}-\sqrt{1+\mu^2\left(\frac{\ell}{2}\right)^2}\right]. \tag{46}$$

These expressions represent the natural parametrization of the classical, that is, inextensible, catenary. This finally proves that, when E approaches infinity, Equations (39), (40), and (41) for the deformable wire converge to Equations (43), (45), and (46) of the classical catenary. A last interesting point concerns the elongation of the deformable wire under the effect of gravity. From Equations (40) and (41), we determine $\left\|\frac{\partial\vec{r}}{\partial s}\right\|$ in the form

$$\left\|\frac{\partial\vec{r}}{\partial s}\right\|=1+\frac{\lambda g}{\mu ES}\sqrt{1+\mu^2\left(s-\frac{\ell}{2}\right)^2}, \tag{47}$$

and therefore the total length ℓ_{def} of the wire, in its equilibrium deformed condition, is

$$\ell_{def}=\ell+\frac{\lambda g}{\mu ES}\int_0^\ell\sqrt{1+\mu^2\left(s-\frac{\ell}{2}\right)^2}ds. \tag{48}$$

Straightforward calculations deliver the total elongation in the form

$$\ell_{def}-\ell=\frac{\lambda g}{\mu^2 ES}\left[\frac{\ell\mu}{2}\sqrt{1+\mu^2\left(\frac{\ell}{2}\right)^2}+\frac{d\mu}{2}-\frac{\lambda g\ell}{2ES}\right]. \tag{49}$$

To conclude, we deduce from Equation (47) that the stretch is always larger than one, proving that the string is always in an elongated state. The elongation is maximum at the constrained ends and minimum at the lower point of the catenary.

Some results showing the behavior of the elastic catenary can be found in Figure 4. First, in the upper left panel, the graphical solution of Equation (39) is observed. The values of μ can be deduced from the intersection between the curves and the horizontal black line. It is observed that these intersections, as the value of Young's modulus increases, converge

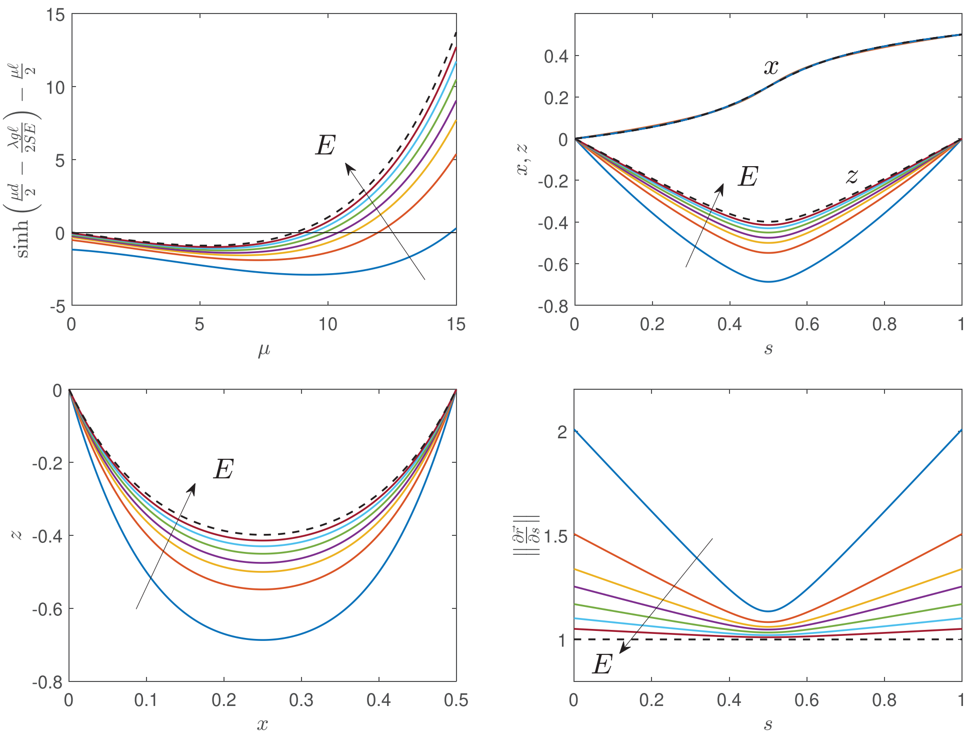

FIGURE 4 Elastic wire anchored at both ends. Left top panel: graphical solution of the transcendental equation given in Equation (39). Right top panel: plot of the parametrization $x = x(s)$, $z = z(s)$. Left bottom panel: shape of the elastic catenary on the plane (x, z). Right bottom panel: plot of the stretch $\left\|\frac{\partial \vec{r}}{\partial s}\right\|$, representing the elongation of the string. In all plots, continuous lines correspond to different values of E, while the black dashed line describes the inextensible catenary ($E \to \infty$). We adopted the parameters $\ell = 1$, $d = 0.5$, $\lambda g = 1$, and $ES = 0.5$, 1, 1.5, 2, 3, 5, 10, $+\infty$, in arbitrary units.

to the value of the parameter μ characteristic of the classical catenary without extensibility. The numerical solution is simply obtained by bisection. In the upper right panel we see the representation of the parametrization $x(s)$, $z(s)$ as a function of the elastic constant E. Note that $x(s)$ is little affected by this parameter, at least in the range considered here. In the lower left panel we show the shape of the elastic catenary, and it can be seen that the total elongation increases as the elasticity decreases, as expected from Equation (49) (the black dashed line corresponds to the classical inextensible catenary). Finally, in the bottom right panel we show the stretch behavior as the elastic constant changes. As proven analytically, the string is always in a stretched state. This corresponds to the fact that the compressional states of the elastic string are always unstable and cannot be observed in solutions at equilibrium. We will see in the following developments that they can be found only when we add the constraint of inextensibility with mass redistribution, while maintaining elasticity.

2.3 | Elastic wire with one end fixed and the other subjected to applied force

We now take into consideration a flexible wire with the first end tethered at $\vec{r}(0) = (0, 0)$ and an arbitrary force $\vec{F} = (F_x, F_z)$ applied at the second end (see Figure 2c). So doing, $\vec{r}(\ell)$ can assume an arbitrary position in the plane (x, z). We need to obtain the boundary condition to use in the end-terminal with the applied force. In order to take into account the force $\vec{F}$, we have to consider into the Lagrangian density in Equation (12) an additional energy density of the form

$$\mathcal{U}_F = -\vec{F} \cdot \vec{r}\delta(s - \ell), \tag{50}$$

where $\delta(\cdot)$ represents the Dirac delta function. Therefore, the following term should be added to the the action given in Equation (13)

$$\mathcal{S}_F = -\int_0^T \int_0^\ell \mathcal{U}_F ds dt. \tag{51}$$

The perturbation of this term corresponds to

$$\mathcal{S}_F(\vec{r}+\alpha\vec{h}) = \int_0^T \int_0^\ell \vec{F}\cdot(\vec{r}+\alpha\vec{h})\delta(s-\ell)dsdt, \tag{52}$$

and its derivative with respect to α is obtained as

$$\frac{\partial \mathcal{S}_F(\vec{r}+\alpha\vec{h})}{\partial\alpha} = \int_0^T \int_0^\ell \vec{F}\cdot\vec{h}\delta(s-\ell)dsdt = \int_0^t \vec{F}\cdot\vec{h}(\ell,t)dt. \tag{53}$$

This term must be added to Equation (17), and then, the spatial boundary conditions can be written as

$$-\int_0^T SE\left(1-\frac{1}{\left\|\frac{\partial\vec{r}}{\partial s}\right\|}\right)\frac{\partial\vec{r}}{\partial s}\cdot\vec{h}dt\Bigg|_0^\ell + \int_0^t \vec{F}\cdot\vec{h}(\ell,t)dt = 0. \tag{54}$$

For $s=0$ (tethered end), the perturbation $\vec{h}$ must be zero and therefore the condition is $\vec{r}(0,t)=(0,0)$, as expected. For $s=\ell$ (end-terminal with the applied force), $\vec{h}$ is free and we get the new boundary condition

$$SE\left(1-\frac{1}{\left\|\frac{\partial\vec{r}}{\partial s}\right\|}\right)\frac{\partial\vec{r}}{\partial s} = \vec{F}. \tag{55}$$

For the static case under investigation, Equation (25) remains valid, and therefore Equation (55) allows us to directly obtain the coefficients C and D as follows

$$C = \frac{F_x}{SE},\quad D = \frac{F_z}{SE} - \frac{\lambda g\ell}{SE}. \tag{56}$$

The differential equations for this problem can be finally written in the form

$$\left[1-\frac{1}{\left\|\frac{\partial\vec{r}}{\partial s}\right\|}\right]\frac{\partial x}{\partial s} = \frac{F_x}{SE},\quad \left[1-\frac{1}{\left\|\frac{\partial\vec{r}}{\partial s}\right\|}\right]\frac{\partial z}{\partial s} = \frac{\lambda g}{SE}(s-\ell) + \frac{F_z}{SE}, \tag{57}$$

and can be solved exactly as in the previous case. Straightforward calculations deliver the solutions

$$x(s) = \frac{F_x}{SE}s + \frac{F_x}{\lambda g}\ln\frac{F_z-\lambda g(\ell-s)+\sqrt{F_x^2+[F_z-\lambda g(\ell-s)]^2}}{F_z-\lambda g\ell+\sqrt{F_x^2+[F_z-\lambda g\ell]^2}}, \tag{58}$$

$$z(s) = \frac{\lambda g}{2SE}s^2 + \frac{F_z-\lambda g\ell}{SE}s + \frac{1}{\lambda g}\left[\sqrt{F_x^2+[F_z-\lambda g(\ell-s)]^2} - \sqrt{F_x^2+[F_z-\lambda g\ell]^2}\right], \tag{59}$$

which are correct for any applied force $\vec{F}=(F_x,F_z)$.

A simple application concerns the case with $F_x=0$ and $F_z=0$. Under these conditions we exactly retrieve the solution given in Equation (23) for the elastic wire hanging from one end. Moreover, the obtained solution can be also applied to study the elastic wire anchored at both ends. Indeed, our solution can be written in the form $x(s)=f(s,\vec{F})$ and $z(s)=g(s,\vec{F})$, where the function f and g are defined in Equations (58) and (59), respectively. To study the problem analyzed in Section 2.2, we have to impose $f(\ell,\vec{F})=d$ and $g(\ell,\vec{F})=0$, where d is the distance between the constrained ends. Once

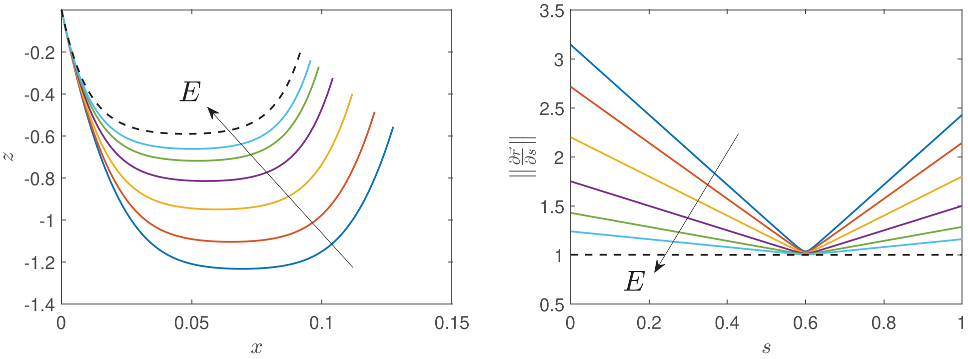

FIGURE 5 Elastic wire with one end fixed and the other subjected to applied force. Left panel: shape of the catenary in the (x, z) plane. Right panel: plot of the stretch $\left\|\frac{\partial \vec{r}}{\partial s}\right\|$, representing the elongation of the string. In both plots, continuous lines correspond to different values of E, while the black dashed line describes the inextensible catenary ($E \to \infty$). We adopted the parameters $\ell = 1$, $\lambda g = 1$, $F_x = 0.01$, $F_z = 0.4$, and $ES = 0.28, 0.35, 0.5, 0.8, 1.4, 2.5, +\infty$, in arbitrary units.

this system of equations is solved with respect to the force $\vec{F}$, applied by the right constraint to the chain, the parametric representation $x(s) = f(s, \vec{F})$, $z(s) = g(s, \vec{F})$ is fully determined. Simple calculations make it possible to find the vertical component of the force $F_z = \lambda g\ell/2$ from $g(\ell, \vec{F}) = 0$. This means, of course, that each end must support half the total weight of the wire. Moreover, from $f(\ell, \vec{F}) = d$, we get the equation for the horizontal component of the force F_x in the form

$$\frac{\lambda g\ell}{2F_x} = \sinh\left(\frac{\lambda g d}{2F_x} - \frac{\lambda g\ell}{2SE}\right). \tag{60}$$

By comparing this equation with Equation (39), we deduce that $\mu = \lambda g/F_x$. This result gives a physical meaning of the parameter μ, which is in fact related with the horizontal component of the force applied to the second end-terminal to impose the positional constraint. We finally underline that, in the inextensible case with $E \to \infty$, Equations (58) and (59) can be easily simplified by obtaining new expressions representing a classical catenary with the first fixed end, and a force applied to the second end. In cases with an applied force in the second end, solutions are always explicit, as opposed to cases with both ends fixed where an equation always remains in implicit form.

An example application of the results found is shown in Figure 5, where the shape of an elastic catenary with a given force applied to the second end is shown (left panel). The various colored curves correspond to different elasticities, while the black dashed curve refers to the inextensible catenary. We observe that the slope of the curve in the right end is independent of the elastic constant since it depends only on the applied force $\vec{F}$. In addition, the curve becomes longer as the elastic constant decreases, as seen previously in other cases. Thus, the more deformable the wire (low Young's modulus), the lower the minimum of the catenary curve. Conversely, the stiffer the wire is, the higher the minimum of the curve will be. Of course, during this process the end of the thread with the applied force shifts so as to minimize the total energy of the system. In the right panel one can see that the string is stretched everywhere and the minimum of stretching is in the minimum of the curve. In this panel of the figure, we see that the stretch is practically linear in the two regions to the left and right of the minimum, and this means that the mass density increases almost linearly from the extreme left to the minimum and decreases almost linearly from the minimum to the extreme right. Let us first observe that the minimum of the curve exists only under certain force conditions: clearly it is present when the F_z component is approximately equal to half the weight of the wire $\lambda g\ell/2$. If the minimum is actually present in the equilibrium configuration, the transition between the two quasistraight behaviors to the left and right of the minimum is sharper when λ is high and when E is small. That is, we can say that the transition is sharper when the ratio $\lambda g/(SE)$ is high. In fact, in this condition the high weight produces a great elongation of the thread, aided by its low elastic constant.

3 | VARIATIONAL APPROACH FOR INEXTENSIBLE STRINGS WITH MASS REDISTRIBUTION

We introduce a rather different model but still based on previous developments. We suppose to consider a chain of masses and springs like the one described in Section 1, but this time we impose overall thread inextensibility. In other words, we can say that masses are always connected to each other by a two-body elastic interaction but can only move on an inextensible but flexible wire of assigned length. After performing the continuum limit of this situation, we obtain a Lagrangian density as in Section 1, but we have to consider an additional constraint given by the fixed length of the wire

$$\int_0^\ell \left\|\frac{\partial \vec{r}}{\partial s}\right\| ds = \ell. \tag{61}$$

We note that this condition fixes the length but the norm, that is, the stretch, will generally not be identically equal to one since there may be regions of some compression and regions of some elongation along the wire. They must compensate for each other but locally the norm is not unitary. This also means that the length is fixed but the representation of the deformed configuration is not the natural one because of these local elongations or compressions. In order to simplify the discussion we study here only the static regime of the inextensible wire with mass redistribution. In addition, to study all possible configurations more easily, we consider a Lagrangian density with a term that takes into account a force applied on the second end of the wire

$$\mathcal{L}(\vec{r}) = -\frac{1}{2}SE\left[\left\|\frac{\partial \vec{r}}{\partial s}\right\| - 1\right]^2 - \mathcal{U}(\vec{r}) + \vec{F}\cdot\vec{r}\delta(s-\ell). \tag{62}$$

The Lagrangian function that must be made stationary is then

$$L = \int_0^\ell \mathcal{L}(\vec{r})ds + \Lambda\left[\int_0^\ell \left\|\frac{\partial \vec{r}}{\partial s}\right\| ds - \ell\right], \tag{63}$$

where Λ is the Lagrangian multiplier corresponding to the length constraint. This corresponds to an isoperimetric problem of the calculus of variations. The perturbation of this Lagrangian function is obtained by substituting $\vec{r}$ with $\vec{r} + \alpha\vec{h}$, as follows

$$L(\vec{r}+\alpha\vec{h}) = \int_0^\ell \left\{-\frac{1}{2}SE\left[\left\|\frac{\partial(\vec{r}+\alpha\vec{h})}{\partial s}\right\| - 1\right]^2 - \mathcal{U}(\vec{r}+\alpha\vec{h}) + \vec{F}\cdot(\vec{r}+\alpha\vec{h})\delta(s-\ell)\right\}ds + \Lambda\left[\int_0^\ell \left\|\frac{\partial(\vec{r}+\alpha\vec{h})}{\partial s}\right\| ds - \ell\right], \tag{64}$$

and the Gateaux derivative is eventually obtained as

$$\left.\frac{\partial L(\vec{r}+\alpha\vec{h})}{\partial \alpha}\right|_{\alpha=0} = \int_0^\ell \left\{SE\frac{\partial}{\partial s}\left[\left(1 - \frac{1+\Theta}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right)\frac{\partial \vec{r}}{\partial s}\right] - \frac{\partial \mathcal{U}}{\partial \vec{r}}\right\}\cdot\vec{h}ds - SE\left(1 - \frac{1+\Theta}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right)\frac{\partial \vec{r}}{\partial s}\cdot\vec{h}\Bigg|_0^\ell + \vec{F}\cdot\vec{h}(\ell), \tag{65}$$

where we introduced the rescaled Lagrangian multiplier Θ, such that $\Lambda = SE\Theta$. By equating the Gateaux derivative to zero, we obtain the following constrained differential problem

$$SE\frac{\partial}{\partial s}\left[\left(1 - \frac{1+\Theta}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right)\frac{\partial \vec{r}}{\partial s}\right] = \frac{\partial \mathcal{U}}{\partial \vec{r}}, \tag{66}$$

$$\int_0^\ell \left\|\frac{\partial \vec{r}}{\partial s}\right\| ds = \ell, \tag{67}$$

where the second equation is useful to obtain the value of Θ. Concerning the boundary conditions, we can have the following cases: (i) if both ends are fixed ($\vec{F}$ is not relevant in this case), we have $\vec{h}(0) = \vec{h}(\ell) = 0$, and the conditions are simply stated as $\vec{r}(0) = \vec{a}$ and $\vec{r}(\ell) = \vec{b}$, where $\vec{a}$ and $\vec{b}$ are given vectors; (ii) when the first end is fixed, say at position $\vec{a}$, and the second end is subjected to a force $\vec{F}$, we have $\vec{r}(0) = \vec{a}$ and

$$SE\left(1 - \frac{1+\Theta}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right)\frac{\partial \vec{r}}{\partial s} = \vec{F}, \text{ if } s = \ell; \tag{68}$$

(iii) finally, if the first end is fixed, say at position $\vec{a}$, and the second end is free, we have $\vec{r}(0) = \vec{a}$ and $\left\|\frac{\partial \vec{r}}{\partial s}\right\| = 1 + \Theta$. In each case, the number of equations is equal to the number of unknown coefficients, so the problem is well posed. Some configurations are studied in the following Sections (see Figure 2 as before).

3.1 Inextensible wire with mass redistribution hanging from one end

We consider a string with the elastic response described in previous Section hanging at the first end, with the second end free (see Figure 2a). The problem is one-dimensional since the string is aligned with the z-axis (for $z < 0$). From Equation (66), where we consider $\mathcal{U} = \lambda g z$, we get the projection on the z-axis

$$SE\frac{\partial}{\partial s}\left\{\left[1 - \frac{1+\Theta}{\left|\frac{\partial z}{\partial s}\right|}\right]\frac{\partial z}{\partial s}\right\} = \lambda g. \tag{69}$$

This equation must be associated with the boundary conditions $z(0) = 0$ (fixed end), $\left|\partial z/\partial s\right| = 1 + \Theta$ for $s = \ell$ (free end), and with the length constraint $\int_0^\ell \left|\partial z/\partial s\right| ds = \ell$. In our case, we can assume that $\partial z/\partial s < 0$. Hence, we get

$$SE\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s} + 1 + \Theta\right) = \lambda g. \tag{70}$$

A first integration yields

$$\frac{\partial z}{\partial s} + 1 + \Theta = \frac{\lambda g}{SE}s + C_1. \tag{71}$$

A second integration leads to

$$z(s) = \frac{\lambda g}{2SE}s^2 - s - s\Theta + C_1 s + C_2. \tag{72}$$

We have to determine the three coefficients C_1, C_2, and Θ. From the condition $z(0) = 0$, we obtain that $C_2 = 0$. Similarly, from the condition $\partial z/\partial s = -1 - \Theta$ for $s = \ell$, we obtain $C_1 = -\lambda g\ell/(SE)$. Since $\left|\partial z/\partial s\right| = 1 + \Theta + \lambda g(\ell - s)/(SE)$, from the condition $\int_0^\ell \left|\partial z/\partial s\right| ds = \ell$, we easily obtain that $\Theta = -\lambda g\ell/(2SE)$. In summary, we have shown that

$$z(s) = -s - \frac{\lambda g}{2SE}s(\ell - s), \tag{73}$$

$$\left|\frac{\partial z}{\partial s}\right| = 1 + \frac{\lambda g}{SE}\left(\frac{\ell}{2} - s\right), \tag{74}$$

where s ranges in $(0, \ell)$. An essential point that needs to be commented on, based on these results, is the behavior of the stretch. We know that it must always be positive to have a smooth curve as a solution. Moreover, it is greater than one where the string is stretched and less than one where it is compressed. First, we see that in this case we have both stretching and compression manifestations because of the fixed length constraint. More importantly, it can be pointed out

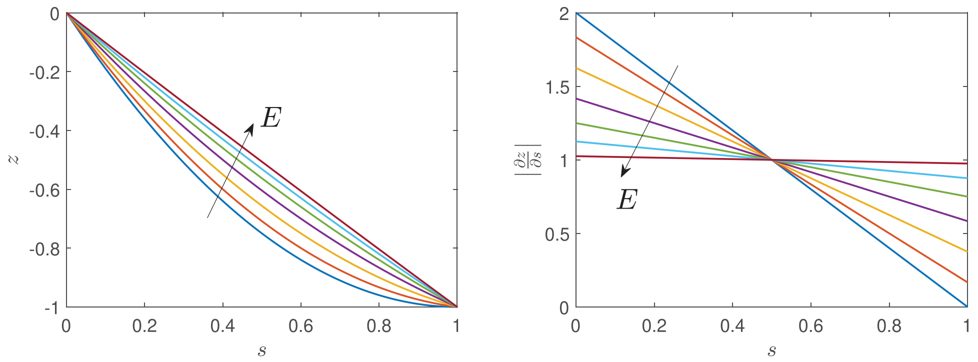

FIGURE 6 Inextensible wire with mass redistribution hanging from one end. Left panel: plot of the parametrization $z = z(s)$ for different values of the elastic constant E. Right panel: plot of the stretch $\left|\frac{\partial z}{\partial s}\right|$, representing the elongation and the compression of the string. We adopted the parameters $\ell = 1$, $\lambda g = 1$, and $ES = 0.5, 0.6, 0.8, 1.2, 2, 4, 20$, in arbitrary units.

that there are values of elastic constant E for which the stretch becomes negative. This means that for these values there are no solutions of the elastic problem that can comply with the geometric isoperimetric constraint. In other words, we can say that for such values of E the system has no solution. It is easily seen that the stretch is always positive, that is, the curve is smooth everywhere, if and only if $SE \geq \lambda g\ell/2$. From the physical point of view, it is observed that for values of SE smaller than this threshold, parts of the string would like to collapse downward, exceeding the length of the string itself, which; however, must be fixed. This incompatibility results in the nonexistence of the solution. This phenomenon can be easily explained by means of a one-dimensional discrete mass-spring chain with the two extremities fixed at $z = 0$ and $z = -\ell$ placed vertically in a gravity field. We prove in the Appendix A that the sequence of masses remains monotonic (and thus the length of the wire remains equal to ℓ) if and only if the previous relation $SE \geq \lambda g\ell/2$ is satisfied. For lower values of the elastic constant, there will be some masses that have an equilibrium position with values of z less than $-\ell$: this is incompatible with the finite length of the wire. This provides an additional interpretation to the threshold value of the elastic constant that allows for solutions to the problem at hand.

These concepts can be observed in Figure 6, where we see the graphical representation of the parametrization $z(s)$, in the left panel. The minimum value of the elastic constant used corresponds to the solution existence threshold, discussed above. We can draw a comparison with the result in Figure 3. Indeed, compared with the elastic string case, in the present situation it is seen that the constraint on length is respected (see left panel), and to do this, there is a compressive state in the lower half of the string (see right panel).

3.2 | Inextensible wire with mass redistribution anchored at both ends

We consider a two-dimensional problem concerning an inextensible wire with the two extremities tethered at $\vec{r}(0) = (0, 0)$ and $\vec{r}(\ell) = (d, 0)$, and subjected to the gravity, similarly to what was considered in Section 2.2 (see Figure 2b). This problem represents the generalization of the inextensible catenary with the additional mass redistribution. From Equation (66), the static equations of the problem are obtained as

$$SE\frac{\partial}{\partial s}\left\{\left[1 - \frac{1+\Theta}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right]\frac{\partial x}{\partial s}\right\} = 0, \qquad SE\frac{\partial}{\partial s}\left\{\left[1 - \frac{1+\Theta}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right]\frac{\partial z}{\partial s}\right\} = \lambda g, \tag{75}$$

and, therefore, a first integration delivers

$$\left[1 - \frac{1+\Theta}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right]\frac{\partial x}{\partial s} = C, \qquad \left[1 - \frac{1+\Theta}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right]\frac{\partial z}{\partial s} = \frac{\lambda g}{SE}s + D, \tag{76}$$

with arbitrary constants C and D. The two differential equations can be analytically integrated by following the steps already developed in Section 2.2, and the results can be found in the integral form

$$x(s) = Cs + (1+\Theta)\int_0^s \frac{dt}{\sqrt{1+\frac{1}{C^2}\left(\frac{\lambda g}{SE}t+D\right)^2}}, \tag{77}$$

$$z(s) = \frac{\lambda g}{2SE}s^2 + Ds + \frac{1+\Theta}{C}\int_0^s \frac{\frac{\lambda g}{SE}t+D}{\sqrt{1+\frac{1}{C^2}\left(\frac{\lambda g}{SE}t+D\right)^2}}dt. \tag{78}$$

As before, the integrals can be calculated in closed form eventually obtaining

$$x(s) = Cs + (1+\theta)C\frac{SE}{\lambda g}\ln\frac{\frac{1}{C}\left(\frac{\lambda g}{SE}s+D\right)+\sqrt{1+\frac{1}{C^2}\left(\frac{\lambda g}{SE}s+D\right)^2}}{\frac{D}{C}+\sqrt{1+\left(\frac{D}{C}\right)^2}}, \tag{79}$$

$$z(s) = \frac{\lambda g}{2SE}s^2 + Ds + (1+\Theta)C\frac{SE}{\lambda g}\left[\sqrt{1+\frac{1}{C^2}\left(\frac{\lambda g}{SE}s+D\right)^2} - \sqrt{1+\left(\frac{D}{C}\right)^2}\right]. \tag{80}$$

These expressions represent the general solution of the problem, independently of the boundary conditions. Indeed, the coefficients C, D, and Θ must be obtained by imposing $x(\ell) = d$, $z(\ell) = 0$, and $\int_0^\ell \left\|\frac{\partial \vec{r}}{\partial s}\right\| ds = \ell$. Then, to impose the fixed length constraint, we need to determine the stretch, that is, the norm $\left\|\frac{\partial \vec{r}}{\partial s}\right\|$. Starting from Equation (76) we directly obtain

$$\left\|\frac{\partial \vec{r}}{\partial s}\right\| = C\sqrt{1+\frac{1}{C^2}\left(\frac{\lambda g}{SE}s+D\right)^2} + 1 + \Theta, \tag{81}$$

and therefore the length of the string can be written as

$$\int_0^\ell \left\|\frac{\partial \vec{r}}{\partial s}\right\| ds = C\int_0^\ell \sqrt{1+\frac{1}{C^2}\left(\frac{\lambda g}{SE}s+D\right)^2}\,ds + (1+\Theta)\ell. \tag{82}$$

By exploiting the classical indefinite integral

$$\int \sqrt{1+\xi^2}d\xi = \frac{1}{2}\xi\sqrt{1+\xi^2} + \frac{1}{2}\ln\left(\xi+\sqrt{1+\xi^2}\right) + const, \tag{83}$$

one easily obtains the constraint on length in the form

$$\Theta = -C^2\frac{SE}{\lambda g\ell}\left[\frac{1}{2C}\left(\frac{\lambda g\ell}{SE}+D\right)\sqrt{1+\frac{1}{C^2}\left(\frac{\lambda g\ell}{SE}+D\right)^2} - \frac{D}{2C}\sqrt{1+\left(\frac{D}{C}\right)^2} + \frac{1}{2}\ln\frac{\frac{1}{C}\left(\frac{\lambda g\ell}{SE}+D\right)+\sqrt{1+\frac{1}{C^2}\left(\frac{\lambda g\ell}{SE}+D\right)^2}}{\frac{D}{C}+\sqrt{1+\left(\frac{D}{C}\right)^2}}\right]. \tag{84}$$

This means that the fixed length condition induces a value of Θ that directly depends on the coefficients C and D, which we now need to find. They can be obtained by fixing $x(\ell) = d$ and $z(\ell) = 0$. The second condition delivers $D = -\lambda g\ell/(2SE)$, as one can easily verify. In addition, the first condition can be written, by introducing $\mu = \lambda g/(CES)$, as

$$d = \frac{\lambda g\ell}{\mu ES} + (1+\theta)\frac{1}{\mu}\ln\frac{\frac{\mu\ell}{2} + \sqrt{1+\left(\frac{\mu\ell}{2}\right)^2}}{-\frac{\mu\ell}{2} + \sqrt{1+\left(\frac{\mu\ell}{2}\right)^2}} = \frac{\lambda g\ell}{\mu ES} + (1+\theta)\frac{2}{\mu}\operatorname{arcsinh}\left(\frac{\mu\ell}{2}\right), \tag{85}$$

or equivalently, as

$$\frac{\mu\ell}{2} = \sinh\left[\frac{\mu d}{2(1+\Theta)} - \frac{\lambda g\ell}{2(1+\Theta)ES}\right], \tag{86}$$

which is a generalization of Equation (39). In this relation the value of Θ must be taken from Equation (84), where we substitute $D = -\lambda g\ell/(2SE)$ and $\mu = \lambda g/(CES)$. We get

$$\Theta = -\frac{1}{\mu^2\ell^2}\frac{\lambda g\ell}{SE}\left[\frac{\mu\ell}{2}\sqrt{1+\left(\frac{\mu\ell}{2}\right)^2} + \frac{1}{2}\ln\frac{\frac{\mu\ell}{2} + \sqrt{1+\left(\frac{\mu\ell}{2}\right)^2}}{-\frac{\mu\ell}{2} + \sqrt{1+\left(\frac{\mu\ell}{2}\right)^2}}\right]. \tag{87}$$

To conclude, we can rewrite the parametric equations of the curve in the final form where we substituted the values of D and μ

$$x(s) = \frac{\lambda g}{\mu ES}s + (1+\theta)\frac{1}{\mu}\ln\frac{\mu\left(s-\frac{\ell}{2}\right) + \sqrt{1+\mu^2\left(s-\frac{\ell}{2}\right)^2}}{-\frac{\mu\ell}{2} + \sqrt{1+\left(\frac{\mu\ell}{2}\right)^2}}, \tag{88}$$

$$z(s) = \frac{\lambda g}{2SE}s^2 - \frac{\lambda g\ell}{2ES}s + (1+\Theta)\frac{1}{\mu}\left[\sqrt{1+\mu^2\left(s-\frac{\ell}{2}\right)^2} - \sqrt{1+\left(\frac{\mu\ell}{2}\right)^2}\right], \tag{89}$$

which represent a generalization of Equations (40) and (41). Here, the value of Θ comes from Equation (87) and the value of μ must be determined by Equation (86), where Θ of course is taken from Equation (87). It means that the transcendental equation for μ is quite complicated and its solution can be only approached through numerical methods.

Some results concerning the elastic catenary with mass redistribution and fixed length can be found in Figure 7. First, in the upper left panel, the graphical solution of Equation (86) combined with Equation (87) is shown for different values of the elastic constant E. The values of μ can be deduced from the intersection between the curves and the horizontal black line. As before, the numerical solution is simply obtained by bisection. The curves seem to have a slightly strange appearance because few of them were drawn so as not to overload the figure. Anyway, the continuity of the family of curves and the convergence to the dashed one can be seen by observing the intersection between the curves and the horizontal black straight line, representing the value of the parameter μ. These intersections represent a monotonically increasing μ value which tends for large values of E to the typical value of the classical catenary. It is important to remark that the classical catenary is re-obtained when the mass remains uniformly distributed in the deformed configuration, and it happens for very large values of the Young's modulus E. Indeed, when E approaches infinity the mass cannot be redistributed over the length of the wire, which is inextensible, and therefore, we retrieve the classical catenary as the solution of the problem. This convergence can be appreciated in Figure 7 (all panels), where the black dashed lines describe the inextensible catenary ($E \to \infty$). In the upper right panel, we see the representation of the parametrization $x(s)$, $z(s)$ as a function of the elastic constant E. Unlike in Figure 4, the $x(s)$-curves (higher up in the figure) now depend significantly on the elastic constant due to mass redistribution. With regard to the $z(s)$-curves (lower down in the figure), it can be seen that the system tends to deform trying to keep the length of the string constant. The bottom left panel shows the shape of the modified catenary and one can see that its length is constant but the shape changes as the elastic constant varies. If this constant is very

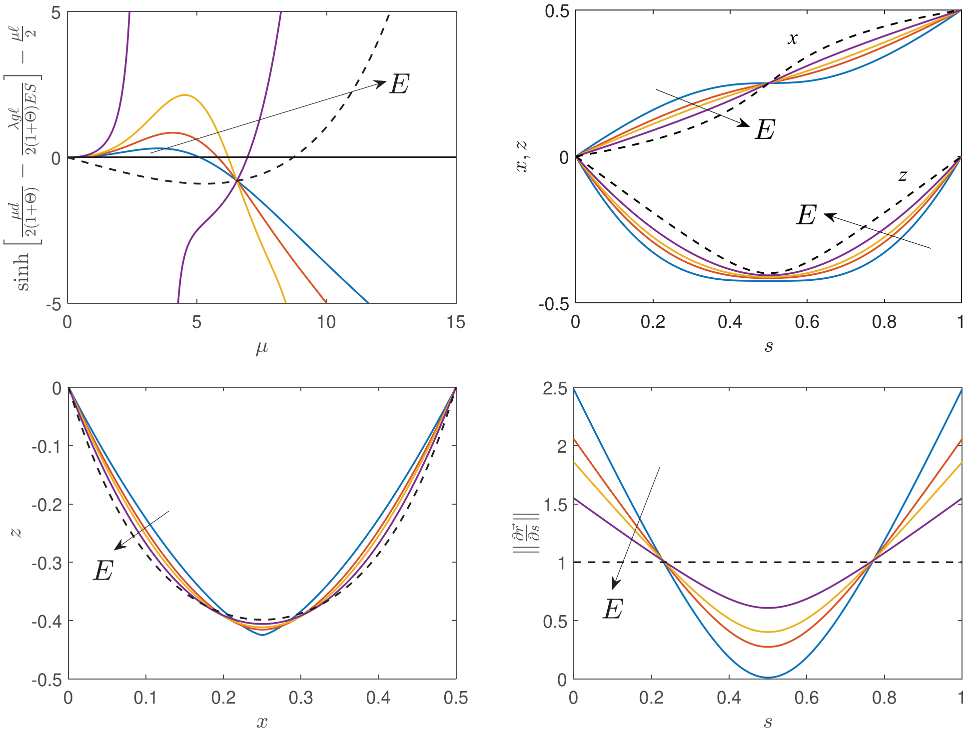

FIGURE 7 Inextensible wire with mass redistribution anchored at both ends: Left top panel: graphical solution of the transcendental equation given in Equation (86). Right top panel: plot of the parametrization $x = x(s)$ (higher up in the panel), $z = z(s)$ (lower down in the panel). Left bottom panel: shape of the elastic catenary on the plane (x, z). Right bottom panel: plot of the stretch $\left\|\frac{\partial\vec{r}}{\partial s}\right\|$, representing the elongation and the compression of the string. In all plots, continuous lines correspond to different values of E, while the black dashed line describes the inextensible catenary ($E \to \infty$). We adopted the parameters $\ell = 1$, $d = 0.5$, $\lambda g = 1$, $ES = 0.138, 0.2, 0.25, 0.4, +\infty$, in arbitrary units.

high, the curve takes on the shape of the classic catenary because the mass remains uniformly distributed along the string (the black dashed line corresponds to the classical inextensible catenary). However, when the elastic constant decreases, the mass can slide more easily towards the minimum of the curve, causing it to change shape: it becomes more pointed and less curved, as one can easily imagine with simple physical interpretations. Finally, in the bottom right panel we show the stretch behavior as the elastic constant changes. As can be seen, the string assumes stretched states in the two highest zones and a compressed state in the lowest zone. This behavior is different from that observed in Figure 4 for the elastic catenary without mass redistribution, for which only the stretched state was present. While the compressional states of the elastic string are always unstable in the case of the elastic string without mass redistribution (because states of compression are compensated by flexibility), in the present case they can be observed because we have introduced the constraint on the fixed length. Nevertheless, such compressional states can also generate the nonexistence of the solution in this case when the elastic constant is too low. In Figure 7, the lowest value used for E is close to the solution existence threshold and in fact the stretch is close to zero at the curve's minimum point. When E becomes even lower, the stretch becomes negative, the curve is no longer regular, and the problem no longer has solution. As already discussed, the incompatibility between the isoperimetric constraint and elastic solution (energy minimization) is at the origin of this nonexistence of the solution (see also the Appendix A).

3.3 | Inextensible wire with mass redistribution with one end fixed and the other subjected to applied force

We consider an inextensible wire with mass redistribution with the first end tethered at the axes origin and with a force $\vec{F}$ applied to the second end (see Figure 2c). The first boundary condition can be therefore written as $x(0) = 0$ and $z(0) = 0$,

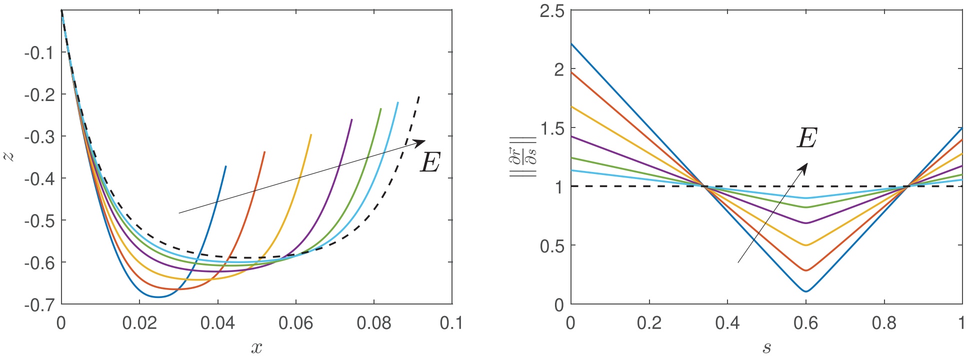

FIGURE 8 Inextensible wire with mass redistribution with one end fixed and the other subjected to applied force. Left panel: shape of the catenary in the (x, z) plane. Right panel: plot of the stretch $\left\|\frac{\partial \vec{r}}{\partial s}\right\|$, representing the elongation and the compression of the string. In both plots, continuous lines correspond to different values of E, while the black dashed line describes the inextensible catenary ($E \to \infty$). We adopted the parameters $\ell = 1$, $\lambda g = 1$, $F_x = 0.01$, $F_z = 0.4$, $ES = 0.28, 0.35, 0.5, 0.8, 1.4, 2.5, +\infty$, in arbitrary units.

and the second one as in Equation (68). This condition can be written explicitly as

$$SE\left[1-\frac{1+\Theta}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right]\frac{\partial x}{\partial s} = F_x, \qquad SE\left[1-\frac{1+\Theta}{\left\|\frac{\partial \vec{r}}{\partial s}\right\|}\right]\frac{\partial z}{\partial s} = F_z, \tag{90}$$

for $s = \ell$, that is, in the right end terminal. Since Equation (76) is still valid for every value of s, we directly determine the values of the constant C and D. We eventually obtain

$$C = \frac{F_x}{SE}, \quad D = \frac{F_z}{SE} - \frac{\lambda g \ell}{SE}. \tag{91}$$

By using these values in Equations (79) and (80), which are evidently also valid here, we get the parametric expressions

$$x(s) = \frac{F_x}{SE}s + (1+\Theta)\frac{F_x}{\lambda g}\ln\frac{F_z - \lambda g(\ell - s) + \sqrt{F_x^2 + [F_z - \lambda g(\ell - s)]^2}}{F_z - \lambda g\ell + \sqrt{F_x^2 + [F_z - \lambda g\ell]^2}}, \tag{92}$$

$$z(s) = \frac{\lambda g}{2SE}s^2 + \frac{F_z - \lambda g\ell}{SE}s + \frac{1+\Theta}{\lambda g}\left[\sqrt{F_x^2 + [F_z - \lambda g(\ell - s)]^2} - \sqrt{F_x^2 + [F_z - \lambda g\ell]^2}\right], \tag{93}$$

describing the generalized catenary in the present case. To conclude, the value of Θ can be obtained by means of Equation (84) with the pertinent values of C and D. The explicit expression follows

$$\Theta = -\frac{1}{2SE\lambda g\ell}\left[F_x^2\ln\frac{F_z + \sqrt{F_x^2 + F_z^2}}{F_z - \lambda g\ell + \sqrt{F_x^2 + [F_z - \lambda g\ell]^2}} + F_z\sqrt{F_x^2 + F_z^2} - (F_z - \lambda g\ell)\sqrt{F_x^2 + [F_z - \lambda g\ell]^2}\right], \tag{94}$$

and must be considered in Equations (92) and (93), to obtain the parametric representation for the solution.

Some results concerning the elastic string with redistribution of mass and fixed length can be seen in Figure 8, in the case where a force is applied to the second end of the string. Here the force is kept constant and the Young's modulus is instead variable in a certain range. In the left panel, we can observe the shape assumed by the string for different values of the elastic constant and with a fixed applied force. The dashed black curve represents the classical inextensible catenary with $E \to \infty$. In the right panel we also show the stretch of the string for the same values of the elastic constant. It is interesting to note that, for the simple elastic catenary whose results are in Figure 5, when the elastic constant decreases, the string is lengthened and consequently the point of force application shifts downward and to the right. In the present case, in Figure 8, the behavior is different: as the elastic constant decreases, the mass concentrates toward the minimum of the curve, forcing the force application point to move leftward and downward. This means that by decreasing the value of the elastic constant of the wire, the mass redistribution phenomenon leads to an accumulation of the mass itself near the center of the wire (i.e., close to the minimum point). This concentration of mass clearly also acts on the right end where the force is applied which moves to the left and even a little downwards. This behavior perfectly characterizes the phenomenon of mass redistribution. In addition, we can observe that in this case in the string there are regions with stretch greater than one, near the ends, and regions with stretch less than one, in the central area. This is the other feature resulting from the constant-length mass redistribution. The lowest value used for the elastic constant results in stretch values close to zero in the minimum of the curve. Again, there is an threshold that defines the minimum value of elastic constant for which the curve remains smooth that is, with positive stretch. As already discussed in Section 2.3, the transition between the two quasistraight behaviors of $\left\|\frac{\partial \vec{r}}{\partial s}\right\|$ to the left and right of the minimum is sharper when the ratio $\lambda g/(SE)$ is high. In fact, in this condition the high weight produces a great elongation of the thread, aided by its low elastic constant.

4 | CONCLUSIONS

In this work, we have developed a variational approach to study specific configurations of deformable strings with diverse physical and geometric characteristics. The objective is to develop these models for application to biological problems, or more broadly, for issues related to soft matter. We have considered a first model that captures the behavior of deformable strings through nonlinear elasticity and a second model that describes deformable strings with mass redistribution while maintaining a constant length. It is important to emphasize that the nonlinearity stems from the finite equilibrium length of the springs utilized in the underlying discrete model. Indeed, the models are derived through the continuous limit of discrete structures. The comparison between the two proposed models enables a deeper exploration of concepts related to the existence and nonexistence of solutions in nonlinear elasticity. To investigate this aspect, we explored various geometries, including scenarios such as the string hanging from one end and free from the other, the string hanging from both ends, and the string suspended from one end with a force applied to the other. On the one hand, for deformable elastic strings, it is observed that compression states cannot exist in equilibrium configurations due to their instability. This instability arises from the inherent complete flexibility of a string. Conversely, in the case of strings with finite length and mass redistribution, both stretched and compressed states can exist in the equilibrium static solutions. These compressed zones, in this instance, are possible owing to the constant length constraint combined with the mass redistribution phenomenon. Specifically, the masses tend to move towards areas with lower potential energy, and in order to adhere to the length constraints, they compress. In this scenario, there may be instances of solutions nonexistence. Specifically, if the elastic constant of the string, which governs the mass redistribution, is too small–falling below a certain threshold–the curve ceases to be regular (or smooth). In such cases, the stretch becomes negative as the solution to the elastic problem becomes incompatible with the fixed length constraint. In practice, the masses attempt to move towards regions with sufficiently low potential energy, causing the failure to satisfy the isoperimetric constraint (see the Appendix A for a simple example based on a one-dimensional discrete system). These phenomena are also evident in the generalization of the catenary with mass redistribution and fixed length. In this geometry, as depicted in the bottom panel on the left of Figure 7, as the elastic constant decreases, the shape becomes more pointed with reduced curvature on both sides. Indeed, the masses tend to slide towards the central area, causing a reshaping of the curve's form. Ideally, in this process, with a hypothetical elastic constant converging to zero, the system should approach a scenario where all the mass concentrates at the minimum of the curve, resulting in a triangular shape. However, due to the dependence of mass distribution on elastic interactions, this extreme situation is unattainable as it generates an incompatibility between inextensibility and energy minimization. Indeed, regular solutions for the shape of the solution are obtained only for values of the elastic constant exceeding a certain stability threshold. In future models, this kind of instability could potentially be mitigated by introducing an additional nonlinearity that prevents the masses from interpenetrating, especially when considering a discrete system for simplicity. Other perspectives include exploring more complex nonlinear elastic responses of the materials used and studying the catenary for strings constrained to an arbitrary surface. It is noteworthy that this latter study can be conducted for both deformable strings and strings with mass redistribution and constant length.

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CONFLICT OF INTEREST STATEMENT

The author declares no potential conflict of interests.

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REFERENCES

[1] Truesdell, C.: The rational mechanics of flexible or elastic bodies 1638–1788. As introduction to Leonhardi Euleri opera omnia, Series II, Volume 11, Part 2, Füssli (1960)
[2] Galilei, G.: Discorsi e Dimostrazioni Matematiche Intorno a Due nuove Scienze Attinenti alla Meccanica e i Movimenti Locali. Giulio Einaudi Editore, Torino (1990)
[3] Simonetti, C.: Galileo e la Catenaria. Archimede, Anno LVIII n°4. Le Monnier, Firenze (2006)
[4] Fahie, J.J.: Galileo, His Life and Work. John Murray, London (1903)
[5] Huygens, C.: Oeuvres Complètes. Nijhoff, The Hague (1888)
[6] Bernoulli, J.: Analysis problematis, de inventione lineae descensus a corpore gravi percurrendae uniformiter. Acta Eruditorum, May, 217–219 (1690)
[7] Mukhopadhyay, U.: Bernoulli brothers, Jacob I and Johann I: A pair of giant mathematicians. Resonance 6(10), 29–37 (2001)
[8] Huygens, C.: Solutio ejusdem problematis. Acta Eruditorum, June, 281–282 (1691)
[9] Bukowski, J.: Christiaan Huygens and the problem of the hanging chain. Coll. Math. J. 39(1), 2–11 (2008)
[10] Bernoulli, J.: Solutio Problematis Funicularii. Acta Eruditorum, June, 274–276 (1691)
[11] Leibniz, G.W.: De linea in quam flexile se pondere proprio curvat. Acta eruditorum, June, 243–247 (1691)
[12] Peiffer, J.: Jacob Bernoulli, teacher and rival of his brother Johann. Electron. Hist. Probab. Stat. 2(1), 1–22 (2006)
[13] Conti, G., Paoletti, R., Trotta, A.: The catenary in history and applications. Sci. Philos. 5(2), 69–94 (2017)
[14] Barnett, J.H.: Enter, stage center: The early drama of the hyperbolic functions. Math. Mag. 77(1), 15–30 (2004)
[15] Euleri, L.: Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes (1744). Reprinted in Leonhardi Euleri Opera Omnia, Series Prima 24, B. G. Teubneri, Leipzig and Berlin (1952)
[16] Euleri, L.: Lntroductio in Analysin Infinitorum, Tomus Primus (1748). Reprinted in Leonhardi Euleri Opera Omnia, Series Prima 8, B. G. Teubneri, Leipzig and Berlin (1922)
[17] Routh, E.J.: A Treatise on Analytical Statics. Cambridge University Press, Cambridge (1891)
[18] Lamb, S.H.: An Elementary Course of Infinitesimal Calculus. Cambridge University Press, Cambridge (1934)
[19] Villaggio, P.: Mathematical Models for Elastic Structures. Cambridge University Press, Cambridge (1997)
[20] Villaggio, P.: Qualitative Methods in Elasticity. Noordhoff International Publishing, Leyden (1977)
[21] Antman, S.S.: Nonlinear problems of elasticity. Applied Mathematical Sciences, vol. 107. Springer Science+Business Media, New York (1995)
[22] Courant, R., Hilbert, D.: Methods of Mathematical Physics. Wiley, New York (1989)
[23] Gelfand, I.M., Fomin, S.V.: Calculus of Variations. Dover Publications, New York (1963)
[24] Elsgoltz, L.: Differential Equations and the Calculus of Variations. MIR Publisher, Moscow (1970)
[25] Yates, R.C.: The catenary and the tractrix. Am. Math. Mon. 66(6), 500–505 (1959)
[26] Cella, P.: Reexamining the catenary. Coll. Math. J. 30(5), 391–393 (1999)
[27] Behroozi, F.: A fresh look at the catenary. Eur. J. Phys. 35, 055007 (2014)
[28] Denzler, J., Hinz, A.: Catenaria vera – the true catenary. Expo. Math. 17, 117–142 (1999)
[29] McIlvaine, G.V.: A new first-principles approach for the catenary. Expo. Math. 37, 333–346 (2019)
[30] Teichman, J., Mahadevan, L.: The viscous catenary. J. Fluid Mech. 478, 71–80 (2003)
[31] Brochard-Wyart, F., de Gennes, P.-G.: The viscous catenary: A poor man's approach. EPL 80, 36001 (2007)
[32] Koulakis, J.P., Mitescu, C.D., Brochard-Wyart, F., De Gennes, P.-G., Guyon, E.: The viscous catenary revisited: experiments and theory. J. Fluid Mech. 609, 87–110 (2008)
[33] Le Merrer, M., Seiwert, J., Quéré, D., Clanet, C.: Shapes of hanging viscous filaments. EPL 84, 56004 (2008)
[34] Zylstra, A., Mitescu, C.: Regime transitions in the viscous catenary. EPL 87, 26003 (2009)
[35] Chakrabarti, B., Hanna, J.A.: Catenaries in viscous fluid. J. Fluids Struct. 66, 490–516 (2016)
[36] Kamrin, K., Mahadevan, L.: Soft catenaries. J. Fluid Mech. 691, 165–177 (2012)
[37] Roy, A., Mahadevan, L., Thiffeault, J.-L.: Fall and rise of a viscoelastic filament. J. Fluid Mech. 563, 283–292 (2006)
[38] Hewitt, I.J, Balmforth, N.J.: Viscoelastic ribbons. J. Fluid Mech. 908, 1–27 (2021)
[39] Buttazzo, G.: Thin structures in elasticity: The variational method. Sem. Mat. Fis. Milano 59, 149–159 (1989)

[40] Acerbi, E., Buttazzo, G., Percivale, D.: A variational definition of the strain energy for an elastic string. J. Elasticity 25, 137–148 (1991)
[41] Antman, S.S.: Multiple equilibrium states of nonlinearly elastic strings. SIAM J. Appl. Math. 37(3), 588–604 (1979)
[42] Wolfe, P.: Multiple equilibrium states of slightly extensible elastic strings. J. Elasticity 47, 205–215 (1997)
[43] Wolfe, P.: The effect of bending stiffness on inextensible cables. Int. J. Eng. Sci. 30(9), 1187–1192 (1992)
[44] Cadelano, E., Giordano, S., Colombo, L.: Interplay between bending and stretching in carbon nanoribbons. Phys. Rev. B 81, 144105 (2010)
[45] Manca, F., Giordano, S., Palla, P.L., Zucca, R., Cleri, F., Colombo, L.: Elasticity of flexible and semiflexible polymers with extensible bonds in the Gibbs and Helmholtz ensembles. J. Chem. Phys. 136, 154906 (2012)
[46] Ahmadi-Kashani, K., Bell, A.J.: The analysis of cables subject to uniformly distributed loads. Eng. Struct. 10, 174–184 (1988)
[47] Teichelmann, G., Schaub, M., Simeon, B.: Modelling and simulation of railway cable systems. ZAMM Z. Angew. Math. Mech. 85(12), 864–877 (2005)
[48] Such, M., Jimenez-Octavio, J.R., Carnicero, A., Lopez-Garcia, O.: An approach based on the catenary equation to deal with static analysis of three dimensional cable structures. Eng. Struct. 31, 2162–2170 (2009)
[49] Greco, L., Impollonia, N., Cuomo, M.: A procedure for the static analysis of cable structures following elastic catenary theory. Int. J. Solids Struct. 51, 1521–1533 (2014)
[50] Zhu, C., Smay, J.E.: Catenary shape evolution of spanning structures in direct-write assembly of colloidal gels. J. Mater. Process. Technol. 212, 727–733 (2012)
[51] Borggräfe, A., Heiligers, J., Ceriotti, M., McInnes, C.R.: Shape control of slack space reflectors using modulated solar pressure. Proc. R. Soc. A 471, 20150119 (2015)
[52] Lepidi, M.: Catenary configuration and geometric stiffness matrix of inextensible cables: Analytical high-order asymptotic solutions for parametric design. Appl. Math. Modell. 128, 1–25 (2024)
[53] Tibert, G.: Numerical Analyses of Cable Roof Structures. Trita-BKN Bulletin, vol. 46. Stockholm (1999)
[54] Irvine, H.M.: Cable Structures. Dover Publications Inc., New York (1992)
[55] Hooke, R.: A Description of Helioscopes, and Some Other Instruments. John Martin, London (1676)
[56] Kemp, M.: Science in Culture. Nature 407, 838 (2000)
[57] Wright, A.: Lucky for some. Nat Phys 5, 8 (2009)
[58] Virga, E.G.: Dissipative shocks in a chain fountain. Phys. Rev. E 89, 053201 (2014)
[59] Biggins, J.S., Warner, M.: Understanding the chain fountain. Proc. R. Soc. A 470, 20130689 (2014)
[60] Biggins, J.S.: Growth and shape of a chain fountain. Eur. Phys. Lett. 106, 44001 (2014)
[61] Hoiczyk, E.: Gliding motility in cyanobacteria: observations and possible explanations. Arch. Microbiol. 174, 11–17 (2000)
[62] Virga, E.G.: Dissipative shocks behind bacteria gliding. Phil. Trans. R. Soc. A 372, 20130360 (2014)
[63] Fujiwara, M., Goh, T., Tsugawa, S., Nakajima, K., Fukaki, H., Fujimoto, K.: Tissue growth constrains root organ outlines into an isometrically scalable shape. Development 148(4), dev196253 (2021)
[64] Guseo, R.: Diffusion of innovations dynamics, biological growth and catenary function. Phys. A: Stat. Mech. Appl. 464, 1–10 (2016)
[65] Vorselen, D., Roos, W.H., MacKintosh, F.C., Wuite, G.J.L., van Loon, J.J.W.A.: The role of the cytoskeleton in sensing changes in gravity by nonspecialized cells. FASEB J. 28(2), 536–547 (2014)
[66] Takahashi, K., Takahashi, H., Furuichi, T., Toyota, M., Furutani-Seiki, M., Kobayashi, T., Watanabe-Takano, H., Shinohara, M., Numaga-Tomita, T., Sakaue-Sawano, A., Miyawaki, A., Naruse, K.: Gravity sensing in plant and animal cells. npj Microgravity 7, 2 (2021)
[67] Singh, M., Gupta, A., Laxmi, A.: Striking the right chord: Signaling enigma during root gravitropism. Front. Plant Sci. 8, 1304 (2017)

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APPENDIX A: DISCRETE SYSTEM EXPLAINING THE POSSIBLE NONEXISTENCE OF SOLUTIONS FOR INEXTENSIBLE STRINGS WITH MASS REDISTRIBUTION

In this Appendix we give a further interpretation to the possible nonexistence of solutions for inextensible strings with mass redistribution. In particular, we refer to the case of an elastic wire hanging from one hand discussed in Section 3.1. We consider the discrete version of the configuration depicted in Figure 2a. We suppose that the vertical wire is composed of N springs with equilibrium length $L = \ell/N$ and elastic constant k. We define $z_0 = 0$, $z_N = -\ell$ and $z_1, z_2, \dots, z_N$ correspond to the position of $N-1$ particles of mass m. It means that the first vertical spring is between $z_0 = 0$ and $z = z_1$, the second one between z_1 and z_2, and finally, the last one between z_{N_1} and $z_N = -\ell = -NL$. The fixed extremities correspond to the inextensibility condition of the string with mass redistribution. Considering the z-axis oriented as in Figure 2, it can be seen that all z_i coordinates will take negative values and that the total potential energy U of the system can easily be written in the form

$$U = \frac{1}{2}k\sum_{i=1}^{N}(z_{i-1} - z_i - L)^2 + mg\sum_{i=1}^{N-1} z_i, \tag{A1}$$

where we considered the contribution of the elastic springs and the contribution of the gravity. The equilibrium of the system can be obtained by minimizing this potential energy. We therefore calculate the partial derivatives

$$\frac{\partial U}{\partial z_j} = k\sum_{i=1}^{N}(z_{i-1} - z_i - L)(\delta_{j,i-1} - \delta_{j,i}) + mg, \tag{A2}$$

which are valid for $k = 1, \ldots, N-1$ and where $\delta_{j,i}$ is the Kronecker's delta symbol. Simplifying and equalizing to zero, we obtain

$$\frac{\partial U}{\partial z_j} = k(-z_{j+1} + 2z_j - z_{j-1}) + mg = 0. \tag{A3}$$

We can now introduce the variables $d_j = z_{j-1} - z_j$, which represent the length of the deformed springs from $j = 1$ to $j = N$. The equilibrium conditions are then reduced to

$$d_j - d_{j+1} = \frac{mg}{k}, \tag{A4}$$

for $k = 1, \ldots, N-1$, with the condition that $\sum_{i=1}^{N} d_i = NL = \ell$. From Equation (A4) we recursively obtain that $d_1 = d_N + (N-1)mg/k$, $d_2 = d_N + (N-2)mg/k$, $d_3 = d_N + (N-3)mg/k$ and so on, such that the general relationship becomes

$$d_j = d_N + (N-j)\frac{mg}{k}, \tag{A5}$$

for $j = 1, \ldots, N-1$. These expressions can be substituted in the total length condition $\sum_{i=1}^{N} d_i = NL = \ell$, yielding

$$Nd_N + \sum_{i=1}^{N-1}(N-i)\frac{mg}{k} = NL. \tag{A6}$$

Recalling that $\sum_{i=1}^{N-1} i = N(N-1)/2$, we obtain the result for d_N

$$d_N = L - \frac{N-1}{2}\frac{mg}{k}. \tag{A7}$$

By using Equation (A5), we obtain all the spring lengths as

$$d_j = L + \left(\frac{N+1}{2} - j\right)\frac{mg}{k}. \tag{A8}$$

All masses are contained in the segment $(-\ell, 0)$, and therefore the inextensibility constraint is satisfied, if and only if all lengths d_j are positive. This is equivalent to saying that d_N is positive, in fact we see that $d_j > d_N$ for all $j = 1, \ldots, N-1$, see Equation (A5). Now, we see that the condition $d_N > 0$ leads to $L > \frac{N-1}{2}\frac{mg}{k}$, where $k = ES/L$ in the continuum limit. It means that the condition becomes $ES > mg(N-1)/2$. To conclude, we observe that $(N-1)m$ represents the total mass of the chain, which can also be written as $\lambda\ell$, and the condition $d_N > 0$ assumes the final form $ES > \lambda\ell g/2$. The obtained condition coincides with the one obtained in Section 3.1 by means of a completely different method based on the variational approach. This clarifies the meaning of this condition. If we use values of Young's modulus below the threshold found, equilibrium for the discrete chain exists but some masses leave the segment $(-\ell, 0)$ not respecting inextensibility. When we approach the continuous problem from the variational point of view, in such cases, solutions do not exist because of the incompatibility between equilibrium search and inextensibility constraint.

The origin of this problem must be sought in the structure of the proposed model. The interaction between two contiguous masses in a discrete model is characterized by a spring (nonlinear for the 2D and 3D cases) that describes the elasticity of the wire. This interaction, however, contrary to physical reality, allows one mass to pass over another along the wire. In other words, the interactions included in the model do not impose the impenetrability of matter. If one were to add such impenetrability into the model, solutions would exist for any value of elastic constant, and for example we could have all masses concentrated in the center of the wire in the case of the catenary with mass redistribution. But the model would become too mathematically difficult and the complexity would probably not be manageable. This means that the existence threshold of the solutions has no physical relevance since in reality the masses cannot freely step over each other along the wire. However, this result is particularly interesting mathematically because it offers an example of the existence or nonexistence of solutions in nonlinear elasticity.